

Chapter 11

Differential Equations

11.1 Introduction

A differential equation is a relationship between some (unknown) function and one of its derivatives. Generally, such equations are encountered in scientific problems in which a statement is made about some rate of change. The goal is usually to identify the function from the given relationship, and a given initial value. The topic of differential equations has been explored previously in the context of differential calculus, in a previous set of notes. In that previous work, we concentrated on differential equations whose solution could be guessed, or whose solution was supplied in advance. We also explored some of the fascinating geometric and qualitative properties of such equations and their predictions.

Now that we have techniques of integration, we revisit the topic of differential equations. We will find that we now have tools that help to find the analytic solution to a variety of simple first-order differential equations (i.e. those involving the first derivative of the unknown function). We will describe the technique of **separation of variables**. This technique works for examples that are simple enough that we can isolate the dependent variable (e.g. y) on one side of the equation, and the independent variable (e.g. time t) on the other side.

11.2 Unlimited population growth

A simple model for population growth

Let $y(t)$ represent the size of a population at time t . We will assume that at time $t = 0$, the population level is specified, i.e. $y(0) = y_0$ is some given constant. We want to find the population at later times, given information about birth and mortality rates, (both of which are here assumed to be constant over time).

The population changes through births and mortality. Suppose that $b > 0$ is the per capita average birth rate, and $m > 0$ the per capita average mortality rate. The assumption that b , m are both constants is a simplification that neglects many biological effects, but will be used for

simplicity in this first example.

The statement that the population increases through births and decreases due to mortality, can be restated as

$$\text{rate of change of } y = \text{rate of births} - \text{rate of mortality}$$

where the rate of births is given by the product of the per capita average birth rate b and the population size y . Similarly, the rate of mortality is given by my . Translating the rate of change into the corresponding derivative of y leads to

$$\frac{dy}{dt} = by - my = (b - m)y.$$

Define

$$k = b - m.$$

Then k is the *net per capita growth rate* of the population. We can distinguish two possible cases: $b > m$ means that there are more births than deaths, so we expect the population to grow. $b < m$ means that there are more deaths than births, so that the population will eventually go extinct. To summarize, this simple model of unlimited growth leads to the differential equation. and initial condition:

$$\frac{dy}{dt} = ky \quad y(0) = y_0.$$

We look for the function $y(t)$ that describes the population size at any future time t , given its initial size at some time $t = 0$.

Separation of variables and integration

Starting with the differential equation

$$\frac{dy}{dt} = ky,$$

we think of dx and dy as “differentials” in the sense defined in a previous chapter. We rearrange and rewrite the above equation now in the form

$$\frac{1}{y} dy = k dt,$$

and integrate over the interval $0 \leq t \leq T$. (During this time interval, the value of y changes from its known initial value y_0 , to some final value that we will refer to as to $y(T)$. (i.e. $y_0 \leq y \leq y(T)$). We do not yet know $y(T)$, but part of our goal is to find that value, i.e to predict the future behaviour of y . Integrating leads to

$$\begin{aligned} \int_{y_0}^{y(T)} \frac{1}{y} dy &= \int_0^T k dt \\ \ln |y| \Big|_{y_0}^{y(T)} &= kt \Big|_0^T \\ \ln |y(T)| - \ln |y_0| &= k(T - 0) \end{aligned}$$

$$\ln \left| \frac{y(T)}{y_0} \right| = kT$$

$$\frac{y(T)}{y_0} = e^{kT}$$

$$y(T) = y_0 e^{kT}.$$

But this result holds for any arbitrary final time, T . In other words, since this is true for any time we chose, we can set $T = t$, arriving at the prediction

$$y(t) = y_0 e^{kt}.$$

The above formula relates the predicted value of y to its initial value, and to all the parameters of the problem. Compare this result to our previous study of such differential equations in the context of exponential functions.

11.3 Terminal velocity and steady states

Let $v(t)$ and $a(t)$ be the velocity and the acceleration, respectively of an object falling under the force of gravity at time t . Suppose that at time $t = 0$, the object starts from rest, i.e. the initial velocity of the object is known to be $v(0) = 0$. When friction is neglected, the object will accelerate,

$$a(t) = g$$

which is equivalent to the statement that the velocity increases at a constant rate,

$$\frac{dv}{dt} = g$$

However, when friction is considered, the object cannot accelerate indefinitely. In fact, a frictional force retards the downwards motion. To a good approximation, that force is proportional to the velocity.

A force balance for the falling object leads to

$$ma(t) = mg - \gamma v(t)$$

where γ is the frictional coefficient. For an object of constant mass, we can divide through by m , so

$$a(t) = g - \frac{\gamma}{m}v(t).$$

Let $k = \gamma/m$. Then, the velocity at any time satisfies the differential equation and initial condition

$$\frac{dv}{dt} = g - kv \quad v(0) = 0.$$

We can find the solution to this differential equation and predict the velocity at any time t using separation of variables.

Consider a time interval $0 \leq t \leq T$, and suppose that, during this time interval, the velocity changes from an initial value of $v(0) = 0$ to the final value, $v(T)$ at the final time, T . Then following our procedure, we write

$$\begin{aligned}\frac{dv}{dt} &= g - kv \\ \frac{dv}{g - kv} &= dt \\ \int_0^{v(T)} \frac{dv}{g - kv} &= \int_0^T dt\end{aligned}$$

Now make the substitution $u = g - kv$. Then $du = -kdv$, $dv = (-1/k)du$, so we get

$$\begin{aligned}\int_{v=0}^{v=v(T)} -\left(\frac{1}{k}\right) \frac{1}{u} du &= \int_0^T dt \\ -\frac{1}{k} \ln |g - kv| \Big|_0^{v(T)} &= t \Big|_0^T\end{aligned}$$

We use the fact that $v(0) = 0$ to write this as

$$\begin{aligned}-\frac{1}{k} (\ln |g - kv(T)| - \ln |g|) &= T \\ -\frac{1}{k} \left(\ln \left| \frac{g - kv(T)}{g} \right| \right) &= T \\ \ln \left| \frac{g - kv(T)}{g} \right| &= -kT.\end{aligned}$$

We can isolate $v(T)$ by exponentiating both sides. The constant g is positive, so we can remove absolute value signs from it.

$$\frac{|g - kv(T)|}{g} = e^{-kT}.$$

To simplify further, we have to consider the sign of the term inside the absolute value in the numerator. In the case we are considering here, with $v(0) = 0$, it will turn out (as we will see shortly) that the quantity $g - kv(T)$ will always be non-negative (i.e. ≥ 0), so that we can write the above as

$$g - kv(T) = ge^{-kT}$$

and finally solve for $v(T)$ to obtain our final result.

$$v(T) = \frac{g}{k}(1 - e^{-kT}).$$

Note that in other cases, if the initial velocity is larger than g/k so that the term in absolute value is negative, we would have to rewrite it as $-(g - kv(T))$ to assure correctness of signs when the absolute value is removed. We will not dwell on these subtleties in our discussion here.)

As before, this formula relating velocity to time holds for any choice of the final time T , so we can write, in general,

$$v(t) = \frac{g}{k}(1 - e^{-kt}).$$

This is the solution to the differential equation for the velocity of the object with friction, given the initial condition $v(0) = 0$. We graph this function in Figure 11.1 to show how it changes with time: We note that as t increases, the term e^{-kt} decreases rapidly, so that the velocity approaches a constant whose value is

$$v(t) \rightarrow \frac{g}{k}.$$

We call this the *terminal velocity*.

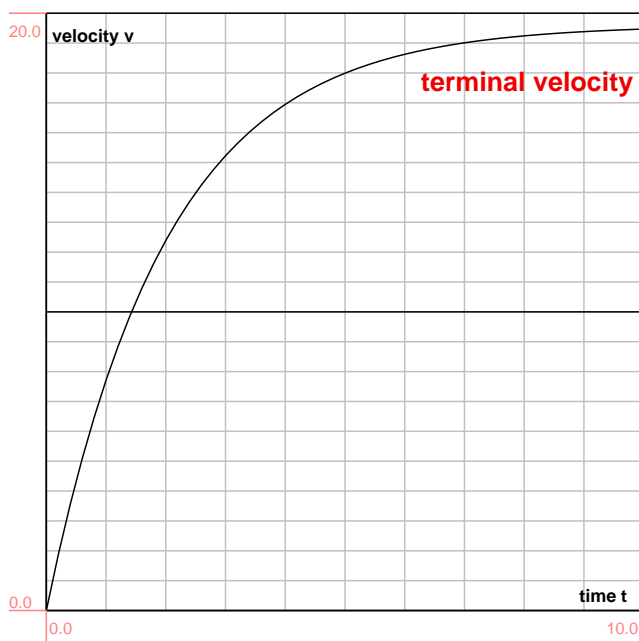


Figure 11.1: The velocity $v(t)$ as a function of time found in Section 11.3. Note that as time increases, the velocity approaches some constant terminal velocity. The parameters used were $g = 9.8 \text{ m/s}^2$ and $k = 0.5$.

Steady state

We might observe that the terminal velocity can also be found quite simply and directly from the *differential equation itself*: it is the **steady state** of the differential equation, i.e. the value for which no further change takes place. The steady state can be found by setting the derivative in the differential equation, to zero, i.e. by letting

$$\frac{dv}{dt} = 0.$$

When this is done, we arrive at

$$g - kv = 0 \quad \Rightarrow \quad v = \frac{g}{k}.$$

Thus, at steady state, the velocity of the falling object is indeed the same as the terminal velocity that we have just discovered.

11.4 Related problems and examples

The example discussed in section 11.3 belongs to a class of problems that share many common features. Generally, this class is represented by differential equations of the form

$$\frac{dy}{dt} = a - by, \quad (11.1)$$

with given initial condition $y(0) = y_0$. Properties of this equation were studied in the context of differential calculus in a previous set of notes. Now, with the same method applied to the problem of terminal velocity, we can integrate this equation by separation of variables, writing

$$\frac{dy}{a - by} = dt$$

and proceeding as in the previous example. We arrive at its solution,

$$y(t) = \frac{a}{b} + \left(y_0 - \frac{a}{b}\right) e^{-bt}. \quad (11.2)$$

The steps are left as an exercise for the reader.

We observe that the steady state of the above equation is given by

$$\frac{dy}{dt} = a - by = 0$$

i.e.

$$y = \frac{a}{b}.$$

Indeed the solution given in the formula 11.2 has the property that as t increases, the exponential term $e^{-bt} \rightarrow 0$ so that the term in large brackets will vanish and $y \rightarrow a/b$. This means that from any initial value, y will approach its steady state level.

This equation has a number of important applications that arise in a variety of context. A few of these are mentioned below.

Blood alcohol

Let $y(t)$ be the level of alcohol in the blood during a night of continual festivity. Suppose that the average rate of drinking is constant. Further assume that alcohol is detoxified in the liver at a rate proportional to its blood level. Then an equation of the form (11.1) would describe the blood level over the period of drinking. $y(0) = 0$ would signify the absence of alcohol in the body at the beginning of the evening. The constant a would reflect the rate of intake per unit volume of the individual's blood: larger people take longer to "get drunk" for a given amount consumed. The constant b would represent the rate of decay of alcohol per unit time due to degradation by the liver; young healthy drinkers have a higher value of b than those who can no longer metabolize alcohol as efficiently.

The solution (11.2) has several features of note: it illustrates the fact that alcohol would increase from the initial level, but only up to a maximum of a/b , where the intake and degradation balance. Indeed, the level $y = a/b$ represents a steady state level (as long as drinking continues).

In the phase of “recovery”, after drinking stops, the above differential equation no longer describes the level of blood alcohol. Instead, the process of recovery is represented by

$$\frac{dy}{dt} = -by, \quad y(0) = y_0.$$

The level of blood alcohol then decays exponentially with rate b from its level at the moment that drinking ends.

Chemical kinetics

The same ideas apply to any chemical substance that is formed at a constant rate (or supplied at a constant rate) a , and then breaks down with rate proportional to its concentration. We then call the constant b the “decay rate constant”.

The variable $y(t)$ represents the concentration of chemical at time t , and the same differential equation describes this chemical process. As above, given any initial level of the substance, $y(t) = y_0$, the level of y will eventually approach the steady state, $y = a/b$.

11.5 Emptying a container

In this section we investigate a new problem in which the differential equation that describes a process will be derived from basic physical principles.

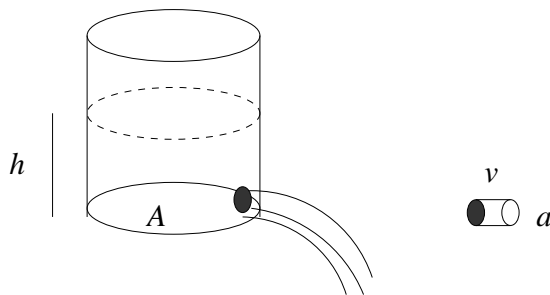


Figure 11.2: We investigate the time it takes to empty a container full of fluid using the techniques of differential equations.

The rate of emptying of a container that has a small hole depends on the height of fluid in the container. One can derive a simple differential equation that describes the rate that the height of the fluid changes using the following physical argument:

Conservation of mass

Suppose that the container is a cylinder, with a constant cross sectional area $A > 0$. Suppose that the area of the hole is a . The rate that fluid leaves through the hole must balance with the rate

that fluid decreases in the container. This principle is called *mass balance*. We will here assume that the density of water is constant, so that we can talk about the net changes in volume (rather than mass).

If we refer to V as the volume of fluid in the container then the rate of change of V is

$$\frac{dV}{dt} = -(\text{rate volume lost as fluid flows out hole}).$$

(The minus sign indicates that the volume is decreasing).

At every second, some amount of fluid leaves through the hole. Suppose we are told that the velocity of the water molecules leaving the hole is precisely $v(t)$. (We will find out how to determine this velocity shortly.) Then in one second, those particles have moved a distance $v \cdot 1 = v$. In fact, all the particles in a little cylinder of length v behind these molecules have also left the hole. Indeed, if we know the area of the hole, we can determine precisely what volume of water exits through the hole each second, namely

$$\text{rate volume lost as fluid flows out hole} = va.$$

(The small inset shows a little "cylindrical unit" of fluid that flows out of the hole per second. The area is a and the length of that little volume is v . Thus the volume leaving per second is va .)

So far we have a relationship between the volume of fluid in the tank and the velocity of the water exiting the hole:

$$\frac{dV}{dt} = -av.$$

Now we need to determine the velocity v of the flow to complete the formulation of the problem.

Conservation of energy

The fluid "picks up speed" because it has "dropped" by a height h from the top of the fluid surface to the hole. In doing so, a small mass of water has simply exchanged some potential energy (due to its relative height above the hole) for kinetic energy (expressed by how fast it is moving). Potential energy of a small mass of water (m) at height h will be mgh , whereas when the water flows out of the hole, its kinetic energy is given by $(1/2)mv^2$ where v is velocity. Thus, for these to balance (so that total energy is conserved) we have

$$\frac{1}{2}mv^2 = mgh.$$

This allows us to relate the velocity of the fluid leaving the hole to the height of the water in the tank, i.e.

$$v^2 = 2gh$$

We have found that the speed of the fluid is related to the height of the water column by the formula

$$v = \sqrt{2gh}.$$

Putting it together

In the last step of our derivation, we convert the statement about rate of change to a differential equation for a single (unknown) function of time. Clearly, we must write all terms in the same units. It proves convenient to express everything in terms of the height of water in the tank, $h(t)$, though we could choose to write the same equation in terms of the volume of fluid in the tank, $V(t)$, if we so desired. Keeping units in an equation consistent is essential. Checking for unit consistency can help to uncover errors in equations, including differential equations.

Recall that the volume of the water in the tank, $V(t)$ is related to the height of fluid $h(t)$ by

$$V(t) = Ah(t),$$

where $A > 0$ is a constant, the cross-sectional area of the tank. Thus, we can simplify as follows:

$$\frac{dV}{dt} = \frac{d(Ah(t))}{dt} = A \frac{d(h(t))}{dt}$$

But

$$\frac{dV}{dt} = -av = -a\sqrt{2gh}.$$

Thus

$$A \frac{d(h(t))}{dt} = -a\sqrt{2gh}.$$

or simply put,

$$\frac{dh}{dt} = -\frac{a}{A}\sqrt{2gh} = -k\sqrt{h}.$$

where k is a constant that depends on the size and shape of the cylinder and its hole:

$$k = \frac{a}{A}\sqrt{2g}.$$

If the area of the hole is very small relative to the cross-sectional area of the tank, then k will be very small, so that the tank will drain very slowly (i.e. the rate of change in h per unit time will not be large). On a planet with a very high gravitational force, the same tank will drain more quickly.

Using the above physical argument which combined simple principles such as conservation of mass and conservation of energy, we have shown that the height $h(t)$ of water in the tank at time t satisfies the following differential equation:

$$\frac{dh}{dt} = -k\sqrt{h}.$$

Clearly, this equation is valid only for h non-negative. Now we will assume a known initial condition, $h(0) = h_0$ and solve the equation by separation of variables.

Solution by separation of variables

We now solve the equation with the given initial condition:

$$\begin{aligned} \frac{dh}{dt} &= -k\sqrt{h} & h(0) &= h_0 \\ \frac{dh}{\sqrt{h}} &= -kdt \\ \int_{h_0}^{h(T)} h^{-1/2} dh &= -k \int_0^T dt \\ \left. \frac{h^{1/2}}{(1/2)} \right|_{h_0}^{h(T)} &= -kT \\ 2 \left(\sqrt{h(T)} - \sqrt{h_0} \right) &= -kT \\ \sqrt{h(T)} &= -k\frac{T}{2} + \sqrt{h_0} \\ h(T) &= \left(\sqrt{h_0} - k\frac{T}{2} \right)^2. \end{aligned}$$

Since this is true for any time t , we can also write the form of the solution as

$$h(t) = \left(\sqrt{h_0} - k\frac{t}{2} \right)^2.$$

The usefulness of this result is that it allows us to predict exactly what fluid height remains in the tank as time goes by. In particular, we can find out when the tank has emptied.

How long will it take the tank to empty?

The tank will be empty when $h(t) = 0$, i.e. when

$$\left(\sqrt{h_0} - k\frac{t}{2} \right)^2 = 0.$$

Solving this equation for the time t , we get

$$k\frac{t}{2} = \sqrt{h_0}$$

or

$$t = \frac{2\sqrt{h_0}}{k}.$$

The time it takes to empty the tank depends on the initial height of water in the tank. But note that the dependence is not simple proportionality! Doubling the height of fluid initially in the tank only increases the time it takes by a factor of $\sqrt{2} \approx 1.41$. Making the hole smaller has a more direct “proportional” effect, since we have found that $k = (a/A)\sqrt{2g}$.

See Figure 11.3 for three solution curves corresponding to different initial heights of $h_0 = 2.5, 5, 10$.

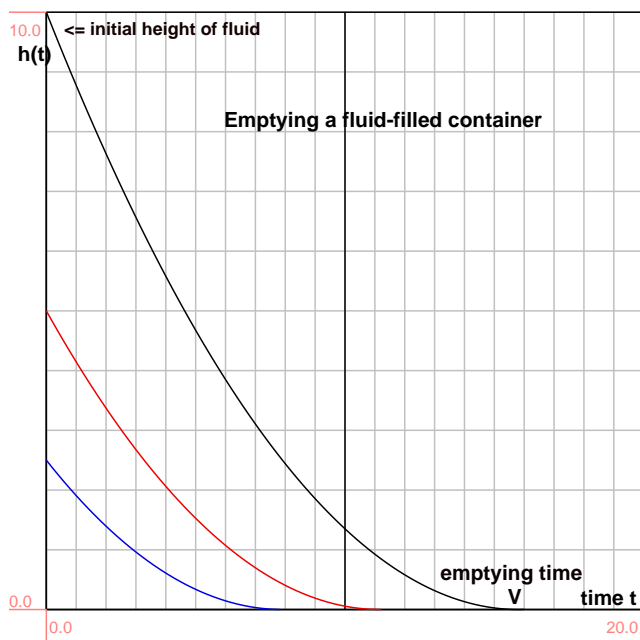


Figure 11.3: Solution curves for three different initial heights $h_0 = 2.5, 5, 10$. of fluid in the container. The parameter $k = 0.4$ in each case.

11.6 Density dependent growth

The simple model discussed in this chapter for population growth has an unrealistic feature of unlimited explosive exponential growth. To correct for this unrealistic feature, a common assumption is that the rate of growth is “density dependent”.

The logistic equation

The logistic equation is the simplest density dependent growth equation, and we study its behaviour below.

Let $N(t)$ be the size of a population at time t . The logistic differential equation states that the rate of change of the population is given by

$$\frac{dN}{dt} = rN \left(\frac{K - N}{K} \right).$$

Here $r > 0$ is called the *intrinsic growth rate* and $K > 0$ is called the *carrying capacity*. K reflects that size of the population that can be sustained by the given environment. We can understand this equation as a modified growth law in which the “density dependent” term, $r(K - N)/K$, replaces the previous constant net growth rate k .

The form of the equation can be simplified if we measure the population in units of the carrying

capacity, instead of “numbers of individuals”. i.e. if we define a new quantity

$$y(t) = \frac{N(t)}{K}.$$

To see this, consider dividing each side of the logistic equation by the constant K . Then

$$\begin{aligned}\frac{dN}{dt} &= rN \left(\frac{K - N}{K} \right). \\ \frac{1}{K} \frac{dN}{dt} &= r \frac{1}{K} N \left(\frac{K - N}{K} \right).\end{aligned}$$

We now group terms conveniently, forming

$$\frac{d(N/K)}{dt} = r(N/K) (1 - (N/K)).$$

Replacing (N/K) by y in each case, we obtain the scaled version of the logistic equation, given by

$$\frac{dy}{dt} = ry(1 - y), \quad y(0) = y_0.$$

We can solve this equation by separation of variables. The idea is essentially the same as our previous examples. To show an alternative method of handling the integration, we will treat both sides as indefinite integrals

$$\begin{aligned}\frac{1}{y(1 - y)} dy &= r dt \\ \int \frac{1}{y(1 - y)} dy &= \int r dt.\end{aligned}$$

The integral on the right will lead to $rt + K$ where K is some constant of integration that we need to incorporate since we do not have endpoints on our integrals. But we must work harder to evaluate the integral on the left. We can do so by partial fractions:

Application of partial fractions

Let

$$I = \int \frac{1}{y(1 - y)} dy$$

Then for some constants A, B we can write

$$I = \int \frac{A}{y} + \frac{B}{1 - y} dy = A \ln |y| - B \ln |1 - y|.$$

(The minus sign in front of B stems from the fact that letting $u = 1 - y$ would lead to $du = -dy$.) We can find A, B from the fact that

$$\frac{A}{y} + \frac{B}{1 - y} = \frac{1}{y(1 - y)}$$

so that

$$A(1 - y) + By = 1.$$

This must be true for all y , and in particular, substituting in $y = 0$ and $y = 1$ leads to $A = 1, B = 1$ so that

$$I = \ln |y| - \ln |1 - y| = \ln \left| \frac{y}{1 - y} \right|.$$

Back to the logistic equation

Returning to the goal of solving the logistic equation, we have

$$\left(\ln \left| \frac{y}{1 - y} \right| \right) = rt + K$$

$$\frac{y(t)}{(1 - y(t))} = e^{rt+K} = e^K e^{rt} = C e^{rt}.$$

In the above step we have simply renamed the constant, e^K by the new name C for simplicity. $C > 0$ is now also an arbitrary constant whose value will be determined from the initial conditions. Indeed, if we substitute $t = 0$ into the most recent equation, we find that

$$\frac{y(0)}{(1 - y(0))} = C e^0 = C$$

so that

$$C = \frac{y_0}{(1 - y_0)}.$$

We will use this fact shortly.

What remains now is some algebra to isolate the desired function $y(t)$

$$y(t) = (1 - y(t))C e^{rt}.$$

$$y(t) (1 + C e^{rt}) = C e^{rt}.$$

$$y(t) = \frac{C e^{rt}}{(1 + C e^{rt})} = \frac{1}{(1/C) e^{-rt} + 1}.$$

The desired function is now expressed in terms of the time t , and the constants r, C . We can also express it in terms of the initial value of y , i.e. y_0 , by using what we know to be true about the constant C , i.e. $C = y_0/(1 - y_0)$. When we do so, we arrive at

$$y(t) = \frac{1}{\frac{1+y_0}{y_0} e^{-rt} + 1} = \frac{y_0}{(y_0 + (1 - y_0) e^{-rt})}.$$

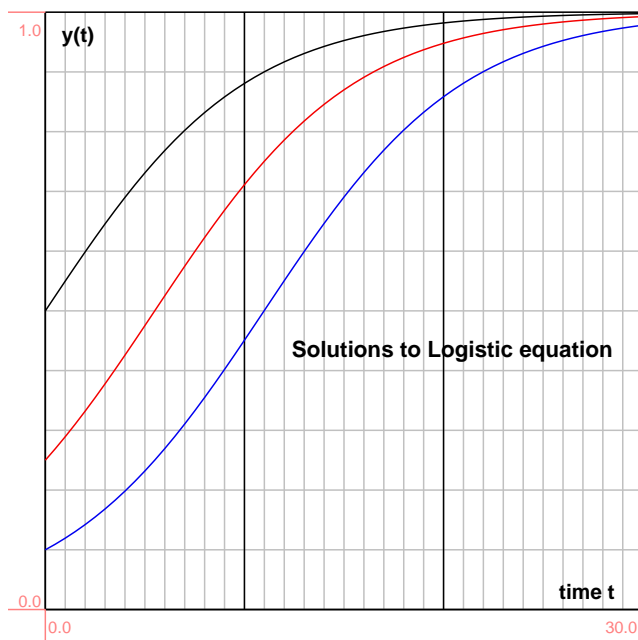


Figure 11.4: Solution curves for three different initial conditions $y_0 = 0.1, 0.25, 0.5$ in the scaled form of the logistic equation.

What this solution tells us

We have arrived at the function that describes population over time as predicted by the logistic equation. The level of population (in units of the carrying capacity K) follows the time-dependent function

$$y(t) = \frac{y_0}{(y_0 + (1 - y_0)e^{-rt})}$$

From this form we observe that after a long time, the term e^{-rt} in the denominator will be negligibly small, and so

$$y(t) \rightarrow \frac{y_0}{y_0} = 1$$

so that y will approach the value 1. This means that

$$(N/K) \rightarrow 1$$

or simply

$$N(t) \rightarrow K.$$

The population will thus settle into a constant level, called a *steady state*, at which no further change will occur.

As an aside, we observe that this too, could have been predicted directly from the differential equation. By setting $dy/dt = 0$, we find that

$$0 = ry(1 - y),$$

which suggests that $y = 1$ (as well as the less interesting case of no population, $y = 0$) is a steady state. Similarly, this could have been found by setting the original logistic differential equation equal to 0:

$$\frac{dN}{dt} = 0$$

to get $N = K$ as the steady state. This reflects that K , the carrying capacity, is the maximum population size that can be sustained by the environment over time.