Chapter 1

Areas, volumes and simple sums

1.1 Introduction

One of the main goals in this course will be calculating areas enclosed by curves in the plane and volumes of three dimensional shapes. We will find that the tools of calculus will provide important and powerful techniques for meeting this goal.

Some shapes are simple enough that no elaborate techniques are needed to compute their areas (or volumes). We briefly survey some of these simple geometric shapes and list what we know or can easily determine about their area or volume.

1.2 Areas of simple shapes

The areas of simple geometrical objects, such as rectangles, parallelograms, triangles, and circles are given by elementary formulae. Indeed, our ability to compute areas and volumes of more elaborate geometrical objects will rest on some of these simple formulae, summarized below.

Rectangular areas

Most integration techniques discussed in this course are based on the idea of carving up irregular shapes into rectangular strips. Thus, areas of rectangles will play an important part in those methods.

- The area of a rectangle with base $b$ and height $h$ is

$$A = b \cdot h$$

- Any parallelogram with height $h$ and base $b$ also has area, $A = b \cdot h$. See Figure 1.1(a) and (b)
Areas of triangular shapes

A few illustrative examples in this chapter will be based on dissecting shapes (such as regular polygons) into triangles. The areas of triangles are easy to compute, and we summarize this review material below. However, triangles will play a less important role in subsequent integration methods.

- The area of a triangle can be obtained by slicing a rectangle or parallelogram in half, as shown in Figure 1.1(c) and (d). Thus, any triangle with base $b$ and height $h$ has area
  \[ A = \frac{1}{2}bh. \]

- In some cases, the height of a triangle is not given, but can be determined from other information provided. For example, if the triangle has sides of length $b$ and $r$ with enclosed angle $\theta$, as shown on Figure 1.1(e) then its height is simply $h = r \sin(\theta)$, and its area is
  \[ A = (1/2)br \sin(\theta) \]

- If the triangle is isosceles, with two sides of equal length, $r$, and base of length $b$, as in Figure 1.1(f) then its height can be obtained from Pythagoras’s theorem, i.e. $h^2 = r^2 - (b/2)^2$ so that the area of the triangle is
  \[ A = (1/2)b\sqrt{r^2 - (b/2)^2}. \]

Example 1: Finding the area of a polygon using triangles

Using the simple ideas reviewed so far, we can determine the areas of more complex geometric shapes. For example, we will compute the area of a regular polygon with $n$ equal sides, where the length of each side is $b = 1$. This will illustrate how triangular regions can be used in the calculation of an area.

Solution

The polygon has $n$ sides, each of length $b = 1$. We dissect the polygon into $n$ isosceles triangles, as shown in Figure 1.2. We do not know the heights of these triangles, but the angle $\theta$ can be found. It is
\[ \theta = 2\pi/n \]

since together, $n$ of these identical angles make up a total of $360^\circ$ or $2\pi$ radians.

Let $h$ stand for the height of one of the triangles in the dissected polygon. Then trigonometric relations relate the height to the base length as follows:
\[ \frac{\text{opp}}{\text{adj}} = \frac{b/2}{h} = \tan(\theta/2) \]
Using the fact that $\theta = 2\pi/n$, and rearranging the above expression, we get

$$h = \frac{b}{2 \tan(\pi/n)}$$

Thus, the area of each of the $n$ triangles is

$$A = \frac{1}{2} bh = \frac{1}{2} b \left( \frac{b}{2 \tan(\pi/n)} \right).$$

The statement of the problem specifies that $b = 1$, so

$$A = \frac{1}{2} \left( \frac{1}{2 \tan(\pi/n)} \right).$$

The area of the entire polygon is then $n$ times this, namely

$$A_{n\text{-gon}} = \frac{n}{4 \tan(\pi/n)}.$$ 

For example, the area of a square (a polygon with 4 equal sides, $n = 4$) is

$$A_{\text{square}} = \frac{4}{4 \tan(\pi/4)} = \frac{1}{\tan(\pi/4)} = 1,$$

where we have used the fact that $\tan(\pi/4) = 1$. 

Figure 1.1: Planar regions whose areas are given by elementary formulae.
Figure 1.2: An \( n \)-sided polygon can be dissected into \( n \) triangles. One of these triangles is shown at right. Since it can be further divided into two Pythagorean triangles, trigonometric relations can be used to find the height \( h \) in terms of the length of the base \( b/2 \) and the angle \( \theta/2 \).

As a second example, the area of a hexagon (6 sided polygon, i.e. \( n = 6 \)) is

\[
A_{\text{hexagon}} = \frac{6}{4 \tan(\pi/6)} = \frac{3}{2(1/\sqrt{3})} = \frac{3\sqrt{3}}{2}.
\]

Here we used the fact that \( \tan(\pi/6) = 1/\sqrt{3} \).

Areas of other shapes

- The area of a circle of radius \( r \) is
  \[
  A = \pi r^2.
  \]
- The surface area of a sphere of radius \( r \) is
  \[
  S_{\text{ball}} = 4\pi r^2.
  \]
- The surface area of a right circular cylinder of height \( h \) and base radius \( r \) is
  \[
  S_{\text{cyl}} = 2\pi rh.
  \]

1.2.1 Example 2: How Archimedes discovered the area of a circle

The formula for the area of a circle of radius \( r \),

\[
A = \pi r^2,
\]

was determined long ago by Archimedes using a clever “dissection” and approximation trick. (The idea actually has interesting parallels with our later development of integration. It involves adding up the areas of a number of simple shapes, in this case triangles, and then taking a limit as that number gets large.)

First, we recall the definition of the constant \( \pi \):
Definition of $\pi$

In any circle, $\pi$ is the ratio of the circumference to the diameter of the circle. (Comment: expressed in terms of the radius, this assertion states the obvious fact that the ratio of $2\pi r$ to $2r$ is $\pi$.)

Shown in Figure 1.3 is a sequence of regular polygons inscribed in the circle. As the number of sides of the polygon increases, its area gradually becomes a better and better approximation of the area inside the circle. Similar observations are central to integral calculus, and we will encounter this idea often. We can compute the area of any one of these polygons by dissecting into triangles. All triangles will be isosceles, since two sides are radii of the circle, whose length we’ll call $r$.

Let $r$ denote the radius of the circle. Suppose that at one stage we have an $n$ sided polygon. (If we knew the side length of that polygon, then we already have a formula for its area. However, this side length is not known to us. Rather, we know that the polygon should fit exactly inside a circle of radius $r$. ) This polygon is made up of $n$ triangles, each one an isosceles triangle with two equal sides of length $r$ and base of undetermined length that we will denote by $b$. (See Figure 1.3.) The area of this triangle is

$$A_{\text{triangle}} = \frac{1}{2}bh.$$ 

The area of the whole polygon, $A_n$, is then

$$A = n \cdot (\text{area of triangle}) = n \frac{1}{2}bh = \frac{1}{2}(nb)h.$$ 

We have grouped terms so that $(nb)$ can be recognized as the perimeter of the polygon (i.e. the sum of the $n$ equal sides of length $b$ each). Now consider what happens when we increase the number of sides of the polygon, taking larger and larger $n$. Then the height of each triangle will get closer to the radius of the circle, and the perimeter of the polygon will get closer and closer to the perimeter of the circle, which is (by definition) $2\pi r$. i.e. as $n \to \infty$,

$$h \to r, \quad (nb) \to 2\pi r.$$
so

\[ A = \frac{1}{2}(nb)h \rightarrow \frac{1}{2}(2\pi r)r = \pi r^2 \]

We have used the notation “→” to mean that in the limit, as \( n \) gets large, the quantity of interest “approaches” the value shown. This argument proves that the area of a circle must be

\[ A = \pi r^2. \]

One of the most important ideas contained in this little argument is that by approximating a shape by a larger and larger number of simple pieces (in this case, a large number of triangles), we get a better and better approximation of its area. This idea will appear again soon, but in most of our standard calculus computations, we will use a collection of rectangles, rather than triangles, to approximate areas of interesting regions in the plane.

**Units**

The units of area can be meters\(^2\) (m\(^2\)), centimeters\(^2\) (cm\(^2\)), square inches, etc.

**1.3 Simple volumes**

![3-dimensional shapes](image)

Figure 1.4: 3-dimensional shapes whose volumes are given by elementary formulae
Later in this course, we will also be computing the volumes of 3D shapes. As in the case of areas, we collect below some basic formulae for volumes of elementary shapes. These will be useful in our later discussions.

1. The volume of a cube of side length \( s \) (Figure 1.4a), is
   \[ V = s^3. \]

2. The volume of a rectangular box of dimensions \( h, w, l \) (Figure 1.4b) is
   \[ V = hwl. \]

3. The volume of a cylinder of base area \( A \) and height \( h \), as in Figure 1.4(c), is
   \[ V = Ah. \]
   This applies for a cylinder with flat base of any shape, circular or not.

4. In particular, the volume of a cylinder with a circular base of radius \( r \), (e.g. a disk) is
   \[ V = h(\pi r^2) \]

5. The volume of a sphere of radius \( r \) (Figure 1.4d), is
   \[ V = \frac{4}{3} \pi r^3. \]

6. The volume of a spherical shell (hollow sphere with a shell of some small thickness, \( T \)) is approximately
   \[ V \approx T \cdot (\text{surface area of sphere}) = 4\pi T r^2. \]

7. Similarly, a cylindrical shell of radius \( r \), height \( h \) and small thickness, \( T \) has volume given approximately by
   \[ V \approx T \cdot (\text{surface area of cylinder}) = 2\pi Trh \]

Units

The units of volume are meters\(^3\) (m\(^3\)), centimeters\(^3\) (cm\(^3\)), cubic inches, etc.

Example 3: Adding up disks: The Tower of Hanoi

In this example, we consider how elementary shapes discussed above can be used to determine volumes of more complex objects. The Tower of Hanoi is a shape consisting of a number of stacked disks. It is a simple calculation to add up the volumes of these disks, but if the tower is large, and comprised of many disks, we would want some shortcut to avoid long sums.

(a) Compute the volume of a tower made up of four disks stacked one on top of the other, as shown in Figure 1.5. Assume that the radii of the disks are 1, 2, 3, 4 units and that each disk has height 1.

(b) Compute the volume of a tower made up of 100 such stacked disks, with radii \( r = 1, 2, \ldots, 99, 100 \).
Figure 1.5: Computing the volume of a set of disks.

Solution

(a) The volume of the four-disk tower is calculated as follows:

\[ V = V_1 + V_2 + V_3 + V_4, \]

where \( V_i \) is the volume of the \( i \)'th disk whose radius is \( r = i, \ i = 1, 2 \ldots 4 \). The height of each disk is \( h = 1 \), so

\[ V = (\pi 1^2) + (\pi 2^2) + (\pi 3^2) + (\pi 4^2) = \pi (1 + 4 + 9 + 16) = 30\pi. \]

(b) The idea will be the same, but we have to calculate

\[ V = \pi (1^2 + 2^2 + 3^2 + \cdots + 99^2 + 100^2). \]

It would be tedious to do this by adding up individual terms, and it is also cumbersome to write down the long list of terms that we will need to add up. This motivates inventing some helpful notation, and finding some clever way of performing such calculations.

1.4 Summations and the “Sigma” notation

We introduce the following notation for the operation of summing a list of numbers:

\[ S = a_1 + a_2 + a_3 + \cdots + a_N \equiv \sum_{k=1}^{N} a_k. \]

The Greek symbol \( \Sigma \) (“Sigma”) indicates summation. The symbol \( k \) used here is called the “index of summation” and it keeps track of where we are in the list of summands. The notation \( k = 1 \) that appears underneath \( \Sigma \) indicates where the sum begins (i.e. which term starts off the series), and the superscript \( N \) tells us where it ends. We will be interested in getting used to this notation, as well as in actually computing the value of the desired sum using a variety of shortcuts.
Example 4a

Suppose we want to form the sum of ten numbers, each equal to 1. We would write this as

\[ S = 1 + 1 + 1 + \ldots + 1 \equiv \sum_{k=1}^{10} 1 \]

The notation \ldots signifies that we have left out some of the terms (out of laziness, or in cases where there are too many to conveniently write down.) We could have just as well written the sum with another symbol (e.g. \( n \)) as the index, i.e. the same operation is implied by

\[ \sum_{n=1}^{10} 1. \]

To compute the value of the sum we use that elementary fact that the sum of ten ones is just 10, so

\[ S = \sum_{k=1}^{10} 1 = 10. \]

Example 4b

Expand and sum the following:

\[ S = \sum_{k=1}^{4} k^2 \]

Solution

\[ S = \sum_{k=1}^{4} k^2 = 1 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30. \]

(We have already seen this sum in part (a) of The Tower of Hanoi.)

Example 4c

Add up the following list of 100 numbers (only a few of them are shown):

\[ S = 3 + 3 + 3 + 3 + \cdots + 3. \]

Solution

There are 100 terms, all equal, so

\[ S = 3 + 3 + 3 + 3 + \cdots + 3 = \sum_{k=1}^{100} 3 = 3 \sum_{k=1}^{100} 1 = 3(100) = 300. \]
Example 4d

Write the following terms in summation notation:

\[ S = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} \]

Solution

We recognize that there is a pattern in the sequence of terms, namely, each one is \(\frac{1}{3}\) raised to an increasing integer power, i.e.

\[ S = \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4. \]

We can represent this with the “Sigma” notation as follows:

\[ S = \sum_{n=1}^{4} \left(\frac{1}{3}\right)^n. \]

The “index” \(n\) starts at 1, and counts up through 2, 3, and 4, while each term has the form of \(\left(\frac{1}{3}\right)^n\). This series is related to a geometric series, to be explored shortly. In most cases, a standard geometric series starts off with the value 1. We can easily modify our notation to include additional terms, for example:

\[ S = \sum_{n=0}^{5} \left(\frac{1}{3}\right)^n = 1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4 + \left(\frac{1}{3}\right)^5. \]

Learning how to compute the sum of such terms will be important to us, and will be described later on in this chapter.

1.4.1 Manipulations of sums

Since addition is commutative and distributive, sums of lists of numbers satisfy many convenient properties. We give a few examples below:

Example 5a

Simplify the following expression:

\[ \sum_{k=1}^{10} 2^k - \sum_{k=3}^{10} 2^k \]
Solution

\[ \sum_{k=1}^{10} 2^k - \sum_{k=3}^{10} 2^k = (2 + 2^2 + 2^3 + \cdots + 2^{10}) - (2^3 + \cdots + 2^{10}) = 2 + 2^2 \]

We could have arrived at this conclusion directly from

\[ \sum_{k=1}^{10} 2^k - \sum_{k=3}^{10} 2^k = \sum_{k=1}^{2} 2^k = 2 + 2^2 = 2 + 4 = 6 \]

The idea is that all but the first two terms in the first sum will cancel. The only remaining terms are those corresponding to \( k = 1 \) and \( k = 2 \).

Example 5b

Expand the following expression:

\[ \sum_{n=0}^{5} (1 + 3^n) \]

Solution

\[ \sum_{n=0}^{5} (1 + 3^n) = \sum_{n=0}^{5} 1 + \sum_{n=0}^{5} 3^n \]

1.5 Summation formulas

In this section we introduce a few examples of useful sums and give formulae that provide a shortcut to dreary calculations.

The sum of consecutive integers (Gauss’ formula)

We first show that the sum of the first \( N \) integers is:

\[ S = 1 + 2 + 3 + \cdots + N = \sum_{k=1}^{N} k = \frac{N(N+1)}{2} \]

The following trick is due to Gauss. By aligning two copies of the above sum, one written backwards, we can easily add them up one by one vertically. We see that:
\[ S = 1 + 2 + \ldots + (N - 1) + N \]
\[ S = N + (N - 1) + \ldots + 2 + 1 \]

The sum is:
\[ 2S = (1 + N) + (1 + N) + \ldots + (1 + N) + (1 + N) \]

Thus, there are \( N \) times the value \( (N + 1) \) above, so that
\[ 2S = N(1 + N), \quad \text{so} \quad S = \frac{N(1 + N)}{2}. \]

Thus, Gauss’ formula is confirmed.

**Example: Adding up the first 1000 integers**

Suppose we want to add up the first 1000 integers. This formula is very useful in what would otherwise be a huge calculation. We find that
\[ S = 1 + 2 + 3 + \cdots + 1000 = \sum_{k=1}^{1000} k = \frac{1000(1 + 1000)}{2} = 500(1001) = 500500. \]

Two other useful formulae are those for the sums of consecutive squares and of consecutive cubes:

**The sum of the first \( N \) consecutive square integers**

\[ S_2 = 1^2 + 2^2 + 3^2 + \cdots + N^2 = \sum_{k=1}^{N} k^2 = \frac{N(N + 1)(2N + 1)}{6} \]

**The sum of the first \( N \) consecutive cube integers**

\[ S_3 = 1^3 + 2^3 + 3^3 + \cdots + N^3 = \sum_{k=1}^{N} k^3 = \left( \frac{N(N + 1)}{2} \right)^2 \]

In the Appendix to this chapter, we show how the formula for the sum of square integers can be proved by a technique called *mathematical induction.*
1.5.1 Back to Hanoi

Armed with the formula for the sum of squares, we can now return to the problem of computing the volume of a tower of 100 stacked disks of heights 1 and radii \( r = 1, 2, \ldots, 99, 100 \). We have

\[
V = \pi (1^2 + 2^2 + 3^2 + \cdots + 99^2 + 100^2) = \pi \sum_{k=1}^{100} k^2 = \pi \frac{100(101)(201)}{6} = 338,350\pi \text{ cubic units.}
\]

1.6 Examples

Compute the following sums:

(a) \( S_a = \sum_{k=1}^{20} (2 - 3k + 2k^2) \)  
(b) \( S_b = \sum_{k=10}^{50} k \)

Solutions

(a) We can separate this into three individual sums, each of which can be handled by algebraic simplification and/or use of the summation formulae developed so far.

\[
S_a = \sum_{k=1}^{20} (2 - 3k + 2k^2) = 2\sum_{k=1}^{20} 1 - 3\sum_{k=1}^{20} k + 2\sum_{k=1}^{20} k^2
\]

Thus, we get

\[
S_a = 2(20) - 3\left(\frac{20(21)}{2}\right) + 2\left(\frac{(20)(21)(41)}{6}\right) = 5150
\]

(b)

\[
S_b = \sum_{k=10}^{50} k
\]

We can express this as a difference of two sums:

\[
S_b = \left(\sum_{k=1}^{50} k\right) - \left(\sum_{k=1}^{9} k\right)
\]

Thus

\[
S_b = \left(\frac{50(51)}{2} - \frac{9(10)}{2}\right) = 1275 - 45 = 1230.
\]

1.7 Summing the geometric series

Consider a sum of terms that all have the form \( r^k \), where \( r \) is some real number and \( k \) is an integer power. We have already seen one example of this type in a previous section. Below we will show that the sum of such a series is given by:
where \( r \neq 1 \). We call this sum a (finite) geometric series. We would like to find an expression for terms of this form in the general case of any real number \( r \), and finite number of terms \( N \). First we note that there are \( N + 1 \) terms in this sum, so that if \( r = 1 \) then

\[
S_N = 1 + 1 + 1 + \ldots + 1 = N + 1
\] (a total of \( N + 1 \) ones added.) If \( r \neq 1 \) we have the following trick:

\[
S = 1 + r + r^2 + \ldots + r^N
\]

\[
-rS = \quad r + r^2 + \ldots + r^{N+1}
\]

Subtracting leads to

\[
S - rS = (1 + r + r^2 + \ldots + r^N) - (r + r^2 + \ldots + r^N + r^{N+1})
\]

Most of the terms on the right hand side cancel, leaving

\[
S(1 - r) = 1 - r^{N+1}
\]

Now dividing both sides by \( 1 - r \) leads to

\[
S = \frac{1 - r^{N+1}}{1 - r}
\]

which was the formula to be established.

Later on, we will be interested in what happens to this sum as the number of terms gets very large. We will then show that, provided \(|r| < 1\), the sum of an infinite geometric series of the above form is

\[
S_N = 1 + r + r^2 + r^3 + \ldots + r^N + \ldots = \sum_{k=0}^{\infty} r^k = \frac{1}{1 - r}.
\]

(This observation stems from the fact that for \(|r| < 1\), the expression \( r^N \) that appears in the sum becomes very small as \( N \) increases.)

### 1.8 Example

Compute the following sum:

\[
S_c = \sum_{k=0}^{10} 2^k
\]
Solution

This is a geometric series

\[ S_c = \sum_{k=0}^{10} 2^k = \frac{1 - 2^{10+1}}{1 - 2} = \frac{1 - 2048}{-1} = 2047. \]

### 1.9 Application of geometric series to the branching structure of the lungs

Our lungs pack an amazingly large surface area into a confined volume. Most of the oxygen exchange takes place in tiny sacs called alveoli at the terminal branches of the airways passages. The bronchial tubes conduct air, and distribute it to the many smaller and smaller tubes that eventually lead to those alveoli. The principle of this efficient organ for oxygen exchange is that these small structures present a very large surface area. Oxygen from the air can diffuse across this area into the bloodstream very efficiently.

The lungs, and many other biological “distribution systems” are composed of a branched structure. The initial segment is quite large. It bifurcates into smaller segments, which then bifurcate further, and so on, resulting in a geometric expansion in the number of branches, their collective volume, length, etc. In this section, we apply geometric series to explore this branched structure of the lung. We will construct a simple mathematical model and explore its consequences. The model will consist in some well-formulated assumptions about the way that “daughter branches” are related to their “parent branch”. Base on these assumptions, and on tools developed in this chapter, we will then predict properties of the structure as a whole. We will be interested particularly in the approximate volume \( V \) and the surface area \( S \) of the airway passages in the lungs.

![Figure 1.6: Air passages in the lungs consist of a branched structure. The index \( n \) refers to the branch generation, starting from the initial segment, labeled 0. All segments are assumed to be cylindrical, with radius \( r_n \) and length \( \ell_n \) in the \( n \)’th generation.](image)
1.9.1 Assumptions

- The airway passages consist of many “generations” of branched segments. We label the largest segment with index “0”, and its daughter segments with index “1”, their successive daughters “2”, and so on down the structure from large to small branch segments. We assume that there are \( M \) “generations”, i.e. the initial segment has undergone \( M \) subdivisions. Figure 1.6 shows only generations 0, 1, and 2. (Typically, for human lungs there can be up to 25-30 generations of branching.)

- At each generation, every segment is approximated as a cylinder of radius \( r_n \) and length \( \ell_n \).

- The number of branches grows along the “tree”. On average, each parent branch produces \( b \) daughter branches. In Figure 1.6, we have illustrated this idea for \( b = 2 \), but in general, averaging over the many branches in the structure \( b \) is smaller than 2. In fact, the rule that links the number of branches in generation \( n \), here denoted \( x_n \) with the number (of smaller branches) in the next generation, \( x_{n+1} \) is

\[
x_{n+1} = bx_n.
\]  

(1.2)

We will assume, for simplicity, that \( b \) is a constant. Since the number of branches is growing down the length of the structure, it must be true that \( b > 1 \). For human lungs, on average, \( 1 < b < 2 \), and we will take \( b \) to be some constant such as \( b = 1.7 \). In actual fact, this simplification cannot be precise, because we have just one segment initially (\( x_0 = 1 \)), and at level 1, the number of branches \( x_1 \) should be some small integer, not a number like “1.7”. However, in many mathematical models, some accuracy is sacrificed to get some quick intuition. Later on, details that were missed and are considered important can be corrected and refined.

- The ratios of radii and lengths of daughters to parents are approximated by “proportional scaling”. This means that the relationship of the radii and lengths satisfy simple rules: The lengths are related by

\[
\ell_{n+1} = \alpha \ell_n,
\]

(1.3)

and the radii are related by

\[
r_{n+1} = \beta r_n
\]

(1.4)

with \( \alpha \) and \( \beta \) positive constants. For example, it could be the case that the radius of daughter branches is 1/2 or 2/3 that of the parent branch. Since the branches decrease in size (while their number grows), we expect that \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \).

Rules such as those given by equations (1.3) and (1.4) are often called self-similar growth laws. We will see in a later chapter, that they are closely linked to the idea of fractals, i.e. theoretical structures produced by iterating such growth laws indefinitely. In a real biological structure, the number of generations is finite. (However, in some cases, a finite geometric series is well-approximated by an infinite sum.)

Actual lungs are not fully symmetric branching structures, but the above approximations are used here for simplicity. According to physiological measurements the scale factors for sizes of daughter to parent size are in the range \( 0.65 \leq \alpha, \beta \leq 0.9 \). (K. G. Horsfield, G. Dart, D. E. Olson, and G. Cumming, (1971) J. Appl. Phys. 31, 207217.) For the purposes of this example, we will use the values of constants given in Table 1.1.
Table 1.1: Typical structure of branched airway passages in lungs.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>radius of first segment</td>
<td>$r_0$</td>
</tr>
<tr>
<td>length of first segment</td>
<td>$\ell_0$</td>
</tr>
<tr>
<td>ratio of daughter to parent length</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>ratio of daughter to parent radius</td>
<td>$\beta$</td>
</tr>
<tr>
<td>number of branch generations</td>
<td>$M$</td>
</tr>
<tr>
<td>average number daughters per parent</td>
<td>$b$</td>
</tr>
</tbody>
</table>

1.9.2 A simple geometric rule

The three equations that govern the rules for successive branching, i.e., equations (1.2), (1.3), and (1.4), are examples of a very generic “geometric progression” recipe. Before returning to the problem at hand, let us examine the implications of this recursive rule, when it is applied to generating the whole structure. Essentially, we will see that the rule linking two generations implies an exponential growth. To see this, let us write out a few first terms in the progression of the sequence $\{x_n\}$:

- initial value: $x_0$
- first iteration: $x_1 = bx_0$
- second iteration: $x_2 = bx_1 = b(bx_0) = b^2x_0$
- third iteration: $x_3 = bx_2 = b(b^2x_0) = b^3x_0$

By the same pattern, at the $n$’th generation, the number of segments will be

$n$’th iteration: $x_n = bx_{n-1} = b(bx_{n-2}) = b(b(bx_{n-3})) = \cdots = (b \cdot b \cdot \cdots b)x_0 = b^n x_0$.

We have arrived at a simple, but important result, namely:

The rule linking two generations, $x_n = bx_{n-1}$ (1.5)

implies that the $n$’th generation will have grown by a factor $b^n$, i.e.,

$x_n = b^n x_0$. (1.6)

This connection between the rule linking two generations and the resulting number of members at each generation is useful in other circumstances. Equation 1.5 is sometimes called a recursion relation, and its solution is given by equation 1.6. We will use the same idea to find the connection between the volumes, and surface areas of successive segments in the branching structure.
1.9.3 Total number of segments

We used the result of section 1.9.2 and the fact that there is one segment in the 0'th generation, i.e. $x_0 = 1$, to conclude that at the $n$'th generation, the number of segments is

$$x_n = x_0 b^n = 1 \cdot b^n = b^n.$$ 

For example, if $b = 2$, the number of segments grows by powers of 2, so that the tree bifurcates with the pattern 1, 2, 4, 8, etc.

To determine how many branch segments there are in total, we add up over all generations, $0, 1, \ldots M$. This is a geometric series, whose sum we can compute. Using equation 1.1, we find

$$N = \sum_{n=0}^{M} b^n = \left( \frac{1 - b^{M+1}}{1 - b} \right).$$ 

Given $b$ and $M$, we can then predict the exact number of segments in the structure. The calculation is summarized further on for values of the branching parameter, $b$, and the number of branch generations, $M$, given in Table 1.1.

1.9.4 Total volume of airways in the lung

Since each lung segment is assumed to be cylindrical, its volume is

$$v_n = \pi r_n^2 \ell_n.$$ 

Here we mean just a single segment in the $n$'th generation of branches. (There are $b^n$ such identical segments in the $n$'th generation, and we will refer to the volume of all of them together as $V_n$ below.)

The length and radius of segments also follow a geometric progression. In fact, the same idea developed above can be used to relate the length and radius of a segment in the $n$'th, generation segment to the length and radius of the original 0'th generation segment, namely,

$$\ell_n = \alpha \ell_{n-1} \Rightarrow \ell_n = \alpha^n \ell_0,$$

and

$$r_n = \beta r_{n-1} \Rightarrow r_n = \beta^n r_0.$$ 

Thus the volume of one segment in generation $n$ is

$$v_n = \pi r_n^2 \ell_n = \pi (\beta^n r_0)^2 (\alpha^n \ell_0) = (\alpha \beta^2)^n (\pi r_0^2 \ell_0).$$ 

This is just a product of the initial segment volume $v_0 = \pi r_0^2 \ell_0$, with the $n$'th power of a certain factor. (That factor takes into account that both the radius and the length are being scaled down at every successive generation of branching.)

$$v_n = (\alpha \beta^2)^n v_0.$$
The total volume of all \( (b^n) \) segments in the \( n \)'th layer is

\[
V_n = b^n v_n = b^n (\alpha \beta^2)^n v_0 = (b \alpha \beta^2)^n v_0.
\]

Here we have grouped terms together to reveal the simple structure of the relationship: one part of the expression is just the initial segment volume, while the other is now a “scale factor” that includes not only changes in length and radius, but also in the number of branches. Letting the constant \( a \) stand for that scale factor, \( a = (b \alpha \beta^2) \) leads to the result that the volume of all segments in the \( n \)'th layer is

\[
V_n = a^n v_0.
\]

The total volume of the structure is obtained by summing the volumes obtained at each layer. Accordingly, total airways volume is

\[
V = v_0 \sum_{n=0}^{30} a^n = v_0 \left( \frac{1 - a^{M+1}}{1 - a} \right).
\]

The similarity of treatment with the previous calculation of number of branches is apparent. We compute the value of the constant \( a \) in Table 1.2, and find the total volume in a summary section below.

### 1.9.5 Total surface area of the lung branches

The surface area of a single segment at generation \( n \), based on its cylindrical shape, is

\[
s_n = 2\pi r_n \ell_n = 2\pi (\beta^n r_0)(\alpha^n \ell_0) = (\alpha \beta)^n \left( \frac{2\pi r_0 \ell_0}{s_0} \right),
\]

where \( s_0 \) is the surface area of the initial segment. Since there are \( b^n \) branches at generation \( n \), the total surface area of all the \( n \)'th generation branches is thus

\[
S_n = b^n (\alpha \beta)^n s_0 = (b \alpha \beta)^n c^n s_0,
\]

where we have let \( c \) stand for the scale factor \( c = (\alpha \beta) \). Thus,

\[
S_n = c^n s_0.
\]

This reveals the similar nature of the problem. To find the total surface area of the airways, we sum up,

\[
S = s_0 \sum_{n=0}^{M} c^n = s_0 \left( \frac{1 - c^{M+1}}{1 - c} \right)
\]

We compute the values of \( s_0 \) and \( c \) in Table 1.2, and summarize final calculations of the total airways surface area in the next section.
Table 1.2: Volume, surface area, scale factors, and other derived quantities. Because \( a \) and \( c \) are bases that will be raised to large powers, it is important to that their values are fairly accurate, so we keep more significant figures.

### 1.9.6 Summary of predictions for specific parameter values

By setting up the model in the above way, we have revealed that each quantity in the structure obeys a simple geometric series, but with distinct “bases” \( b, a \) and \( c \) and coefficients \( 1, v_0, \) and \( s_0 \). This approach has shown that the formula for geometric series applies in each case. Now it remains to merely “plug in” the appropriate quantities. In this section, we collect our results, use the sample values for a model “human lung” given in Table 1.1, or the resulting derived scale factors and quantities in Table 1.2 to finish the task at hand.

#### Total number of segments

\[
N = \sum_{n=0}^{M} b^n = \left( \frac{1 - b^{M+1}}{1 - b} \right) = \left( \frac{1 - (1.7)^{31}}{1 - 1.7} \right) = 1.9898 \times 10^7 \approx 2 \times 10^7
\]

According to this calculation, there are a total of about 20 million branch segments overall (including all layers, form top to bottom) in the entire structure!

#### Total volume of airways

Using the values for \( a \) and \( v_0 \) computed in Table 1.2, we find that the total volume of all segments in the \( n \)'th generation is

\[
V = v_0 \sum_{n=0}^{30} a^n = v_0 \left( \frac{1 - a^{M+1}}{1 - a} \right) = 4.4 \frac{(1 - (1.131588^{31})}{(1 - 1.131588)} = 1510.3 \text{ cm}^3.
\]

Recall that 1 litre = 1000 cm\(^3\). Then we have found that the lung airways contain about 1.5 litres.

#### Total surface area of airways

Using the values of \( s_0 \) and \( c \) in Table 1.2, the total surface area of the tubes that make up the airways is

\[
S = s_0 \sum_{n=0}^{M} c^n = s_0 \left( \frac{1 - c^{M+1}}{1 - c} \right) = 17.6 \frac{(1 - (1.3158^{31})}{(1 - 1.3158)} = 2.76 \times 10^5 \text{ cm}^2
\]
There are 100 cm per meter, and $(100)^2 = 10^4 \text{ cm}^2 \text{ per m}^2$. Thus, the area we have computed is equivalent to about 28 square meters!

### 1.9.7 Exploring the problem numerically

Up to now, all calculations were done using the formulae developed for geometric series. However, sometimes it is more convenient to devise a computer algorithm to implement “rules” and perform repetitive calculations in a problem such as discussed here. The advantage of that approach is that it eliminates tedious calculations by hand, and, in cases where summation formulae are not know to us, reduces the need for analytical computations. It can also provide a shortcut to visual summary of the results. The disadvantage is that it can be less obvious how each of the values of parameters assigned to the problem affects the final answers.

A spreadsheet is an ideal tool for exploring iterated rules such as those given in the lung branching problem. In Figure 1.7 we show the volumes and surface areas associated with the lung airways for parameter values discussed above. Both layer by layer values and cumulative sums leading to total volume and surface area are shown in each of (a) and (c). In (b) and (d), we compare these results to similar graphs in the case that one parameter, the branching number, $b$ is adjusted from 1.7 (original value) to 2. The contrast between the graphs shows how such a small change in this parameter can significantly affect the results.

### 1.9.8 For further independent study

The following problems can be used for further independent exploration of these ideas.

1. In our model, we have assumed that, on average, a parent branch has only “1.7” daughter branches, i.e. that $b = 1.7$. Suppose we had assumed that $b = 2$. What would the total volume $V$ be in that case, keeping all other parameters the same? Explain why this is biologically impossible in the case $M = 30$ generations. For what value of $M$ would $b = 2$ lead to a reasonable result?

2. Suppose that the first 5 generations of branching produce 2 daughters each, but then from generation 6 on, the branching number is $b = 1.7$. How would you set up this variant of the model? How would this affect the calculated volume?

3. In the problem we explored, the net volume and surface area keep growing by larger and larger increments at each “generation” of branching. We would describe this as “unbounded growth”. Explain why this is the case, paying particular attention to the scale factors $a$ and $c$.

4. A problem that the lung structure is “solving” biologically is how to produce a large surface area inside the relatively small volume of a human chest. Ideally, the volume of the lung should not grow in an unbounded way down the structure - i.e. the volume increments that are added every time a new layer of branches is added should get smaller. At the same time, the total surface area should get bigger and bigger. Which single factor or parameter
Figure 1.7: (a) $V_n$, the volume of layer $n$ (red bars), and the cumulative volume down to layer $n$ (yellow bars) are shown for parameters given in Table 1.1. (b) Same as (a) but assuming that parent segments always produce two daughter branches (i.e. $b = 2$). The graphs in (a) and (b) are shown on the same scale to accentuate the much more dramatic growth in (b). (c) and (d): same idea showing the surface area of $n$’th layer (green) and the cumulative surface area to layer $n$ (blue) for original parameters (in c), as well as for the value $b = 2$ (in d).
should we change (and how should we change it) to correct this feature of the model, i.e. to predict that the lung volume remains roughly constant while the surface area increases over the generations.

5. Determine how the branching properties of real human lungs differs from our assumed model, and use similar ideas to refine and correct our estimates. You may want to investigate what is known about the actual branching parameter \( b \), the number of generations of branches, \( M \), and the ratios of lengths and radii that we have assumed. Alternately, you may wish to find parameters for other species and do a comparative study of lungs in a variety of animal sizes.

6. Branching structures are ubiquitous in biology. Many species of plants are based on a regular geometric sequence of branching. Consider a tree that trifurcates (i.e. produces 3 new daughter branches per parent branch, \( b = 3 \)). Explain (a) What biological problem is to be solved in creating such a structure (b) What sorts of constraints must be satisfied by the branching parameters to lead to a viable structure. This is an open-ended problem.

### 1.10 Summary

In this chapter, we collected useful formulae for areas and volumes of simple 2 and 3D shapes. A summary of the most important ones is given in this section. Table 1.3 lists the areas of simple shapes, Table 1.4 the volumes and Table 1.5 the surface areas of 3D shapes.

We used areas of triangles to compute areas of more complicated shapes, including regular polygons. We used a polygon with \( N \) sides to approximate the area of a circle, and then, by letting \( N \) go to infinity, we were able to prove that the area of a circle of radius \( r \) is \( A = \pi r^2 \). This idea, and others related to it, will form a deep underlying theme in the next two chapters and later on in this course.

Finally, introduced some notation for series and collected useful formulae for summation of such series. These are summarized in Table 1.6. We will use these extensively in our next chapter.

<table>
<thead>
<tr>
<th>Object</th>
<th>dimensions</th>
<th>area, ( A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>triangle</td>
<td>base ( b ), height ( h )</td>
<td>( \frac{1}{2}bh )</td>
</tr>
<tr>
<td>rectangle</td>
<td>base ( b ), height ( h )</td>
<td>( bh )</td>
</tr>
<tr>
<td>circle</td>
<td>radius ( r )</td>
<td>( \pi r^2 )</td>
</tr>
</tbody>
</table>

Table 1.3: **Areas of planar regions**
<table>
<thead>
<tr>
<th>Object</th>
<th>dimensions</th>
<th>volume, $V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>box</td>
<td>base $b$, height $h$, width $w$</td>
<td>$hwb$</td>
</tr>
<tr>
<td>circular cylinder</td>
<td>radius $r$, height $h$</td>
<td>$\pi r^2 h$</td>
</tr>
<tr>
<td>sphere</td>
<td>radius $r$</td>
<td>$\frac{4}{3} \pi r^3$</td>
</tr>
<tr>
<td>cylindrical shell*</td>
<td>radius $r$, height $h$, thickness $T$</td>
<td>$2\pi rhT$</td>
</tr>
<tr>
<td>spherical shell*</td>
<td>radius $r$, thickness $T$</td>
<td>$4\pi r^2 T$</td>
</tr>
</tbody>
</table>

Table 1.4: Volumes of 3D shapes. * Assumes a thin shell, i.e. small $T$.

<table>
<thead>
<tr>
<th>Object</th>
<th>dimensions</th>
<th>surface area, $S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>box</td>
<td>base $b$, height $h$, width $w$</td>
<td>$2(bh+bw+hw)$</td>
</tr>
<tr>
<td>circular cylinder</td>
<td>radius $r$, height $h$</td>
<td>$2\pi rh$</td>
</tr>
<tr>
<td>sphere</td>
<td>radius $r$</td>
<td>$4\pi r^2$</td>
</tr>
</tbody>
</table>

Table 1.5: Surface areas of 3D shapes

<table>
<thead>
<tr>
<th>Sum</th>
<th>Notation</th>
<th>Formula</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 + 2 + 3 + \cdots + N$</td>
<td>$\sum_{k=1}^{N} k$</td>
<td>$\frac{N(1+N)}{2}$</td>
<td>Gauss’ formula</td>
</tr>
<tr>
<td>$1^2 + 2^2 + 3^2 + \cdots + N^2$</td>
<td>$\sum_{k=1}^{N} k^2$</td>
<td>$\frac{N(N+1)(2N+1)}{6}$</td>
<td>Sum of squares</td>
</tr>
<tr>
<td>$1^3 + 2^3 + 3^3 + \cdots + N^3$</td>
<td>$\sum_{k=1}^{N} k^3$</td>
<td>$\left( \frac{N(N+1)}{2} \right)^2$</td>
<td>Sum of cubes</td>
</tr>
<tr>
<td>$1 + r + r^2 + r^3 \ldots r^N$</td>
<td>$\sum_{k=0}^{N} r^k$</td>
<td>$\frac{1-r^{N+1}}{1-r}$</td>
<td>Geometric sum</td>
</tr>
</tbody>
</table>

Table 1.6: Useful summation formulae.
1.10.1 Appendix: How to prove the formulae for sums of squares and cubes

Proof by induction (optional)

In this Appendix, we prove the formula for the sum of square integers,

\[ \sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6} \]

using a technique called induction. The idea of the method is to check that the formula works for one or two simple cases (e.g. the “sum” of just one or just two terms of the series), and then show that whenever it works for one case (summing up to \( N \)), it has to also work for the next case (summing up to \( N + 1 \)).

First, we verify that this formula works for a few test cases:

\( N = 1 \): If there is only one term, then clearly, by inspection,

\[ \sum_{k=1}^{1} k^2 = 1^2 = 1 \]

The formula indicates that we should get

\[ S = \frac{1(1+1)(2 \cdot 1 + 1)}{6} = \frac{1(3)}{6} = 1 \]

so this case agrees with the prediction.

\( N = 2 \):

\[ \sum_{k=1}^{2} k^2 = 1^2 + 2^2 = 1 + 4 = 5 \]

The formula would then predict that

\[ S = \frac{2(2+1)(2 \cdot 2 + 1)}{6} = \frac{2(3)(5)}{6} = 5. \]

So far, elementary computation matches with the result predicted by the formula. Now we show that if this formula holds for any one case, e.g. for the sum of the first \( N \) squares, then it is also true for the next case, i.e. for the sum of \( N + 1 \) squares. So we will assume that someone has checked that for some particular value of \( N \) it is true that

\[ S_N = \sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6} \]

Now the sum of the first \( N + 1 \) squares will be just a bit bigger: it will have one more term added to it:

\[ S_{N+1} = \sum_{k=1}^{N+1} k^2 = \sum_{k=1}^{N} k^2 + (N + 1)^2 \]
Thus

\[ S_{N+1} = \frac{N(N + 1)(2N + 1)}{6} + (N + 1)^2. \]

Combining terms, we get

\[ S_{N+1} = (N + 1) \left[ \frac{N(2N + 1)}{6} + (N + 1) \right] \]

\[ S_{N+1} = (N + 1) \frac{2N^2 + N + 6N + 6}{6} = (N + 1) \frac{2N^2 + 7N + 6}{6} \]

Simplifying and factoring the last term leads to

\[ S_{N+1} = (N + 1) \frac{(2N + 3)(N + 2)}{6}. \]

We want to check that this still agrees with what the formula predicts. To make the notation simpler, we will let \( M \) stand for \( N + 1 \). Then, expressing the result in terms of the quantity \( M = N + 1 \) we get

\[ S_M = \sum_{k=1}^{M} k^2 = (N + 1) \frac{2(N + 1) + 1][(N + 1) + 1] = M \frac{2M + 1}{6}. \]

This is the same formula as we started with, only written in terms of \( M \) instead of \( N \). Thus we have verified that the formula works. By *Mathematical Induction* we find that the result has been proved.

**Another method using a trick\(^1\)**

There is another method for determining the sums \( \sum_{k=1}^{n} k^2 \) or \( \sum_{k=1}^{n} k^3 \). Write

\[ (k + 1)^3 - (k - 1)^3 = 6k^2 + 2 \]

so

\[ \sum_{k=1}^{n} ((k + 1)^3 - (k - 1)^3) = \sum_{k=0}^{n} (6k^2 + 2). \]

But looking more carefully at the LHS, we see that

\[ \sum_{k=1}^{n} ((k + 1)^3 - (k - 1)^3) = 2^3 - 0^3 + 3^3 - 1^3 + 4^3 - 2^3 + 5^3 - 3^3 ... + (n + 1)^3 - (n - 1)^3 \]

most of the terms cancel, leaving only \(-1 + n^3 + (n + 1)^3\), so this means that

\[ -1 + n^3 + (n + 1)^3 = 6 \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2 \]

---

\(^1\)I want to thank Robert Israel for contributing this material
so
\[
\sum_{k=1}^{n} k^2 = (-1 + n^3 + (n + 1)^3 - 2n)/6 = (2n^3 + 3n^2 + n)/6
\]

Similarly, the formula for \(\sum_{k=1}^{n} k^3\), can be obtained by starting with
\[
(k + 1)^4 - (k - 1)^4 = 4k^3 + 4k
\]