Chapter 12

Infinite series, improper integrals, and Taylor series

12.1

Determine which of the following sequences converge or diverge

(a) $\{e^n\}$ (b) $\{2^{-n}\}$ (c) $\{ne^{-2n}\}$

(d) $\{\frac{2}{n}\}$ (e) $\{\frac{n}{2}\}$ (f) $\{\ln(n)\}$

12.2

Which of the following series converge and which diverge? Give a reason.

(a) $\sum 1^{-n}$ (b) $\sum 3^{-n}$ (c) $\sum \frac{3}{n}$

(d) $\sum 0.1^n$ (e) $\sum 0.1^{-n}$ (f) $\sum \left(\frac{3}{n}\right)^2$

12.3

Using the spreadsheet or a hand-drawn sketch, compare the areas represented by the following integral and series.

$$\int_1^\infty \frac{1}{x^2}\, dx, \qquad \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Draw your sketch in such a way that the convergence of the integral will imply convergence of the series. (Hint: what kind of comparison should be set up? This will determine how you sketch the bar graph representing the series alongside the function $y = f(x) = 1/x^2$

12.4

Use the spreadsheet to show the first 100 terms in the alternating harmonic series,

$$S = \sum_{k=1}^{100} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots .$$

(You can use a bar graph to represent these terms.) On the same graph, show the first 100 "partial sums", i.e. the graph of

$$S_n = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k}.$$

To what value does this alternating harmonic series converge?

12.5

Use the spreadsheet to show the first 100 terms in the series,

$$S = \sum_{k=1}^{100} (-1)^{k+1} \frac{1}{(2k-1)} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots .$$

(You can use a bar graph to represent these terms.) On the same graph, show the first 100 "partial sums" multiplied by 4, i.e. the graph of

$$4(S_n) = 4 \left(\sum_{k=1}^{n} (-1)^{k+1} \frac{1}{(2k-1)} \right).$$

Show that this converges to π.

Find how many correct decimal places are achieved by adding up the first thousand terms of this series.

12.6

The trunk of a tree gets wider by adding a new growth every year. (In cross section, these show up as "rings", one ring for each year). The first year the trunk has radius 1 inch. In the second year the radius increases by 1/2 inch, in the third year by 1/4 inch, and so on every year. What is the trunk radius for a 20 year old tree? For a 50 year old tree? How big can the radius of the trunk get if the tree keeps growing this way indefinitely? What will the cross-sectional area of the trunk be after a long time?

12.7

A symmetric flat snowflake starts out as a square of size 1 millimeter. Every time step a new branch is added on the furthest edge of each of the previous branches. Each branch is a little square whose area is $1/f$ times the area of its "parent branch". (See Figure 12.1.) What is the area of the snowflake after a very long time? For what values of f is this final area finite? What is the final area in the case that $f = 4$?

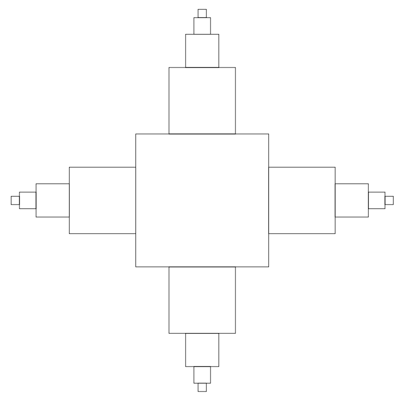

Figure 12.1: A symmetrical snowflake, for problem 12.7

12.8

Consider the integral

$$A = \int_1^D \frac{1}{x^p}\,dx$$

(a) Sketch a region in the plane whose area represents this in each of the following cases:

(i) If $p > 1$ and (ii) $p < 1$.

(b) Evaluate the integral for $p \neq 1$.

(c) How does the area A depend on the value of D in each of the cases (i) and (ii). Does the area increase without bound as D increases ? Or does the area approach some constant ?

(d) With this in mind, how might we try to understand an integral of the form

$$\int_1^\infty \frac{1}{x^p}\,dx$$

12.9

Which of the following improper integrals converge? Give a reason in each case.

(a) $\int_1^\infty \frac{1}{x^{1.001}} dx$ (b) $\int_1^\infty x dx$ (c) $\int_1^\infty x^{-3} dx$

(d) $\int_0^\infty e^x dx$ (e) $\int_0^\infty e^{-2x} dx$ (f) $\int_0^\infty xe^{-x} dx$

12.10

The gravitational force between two objects of mass m_1 and m_2 is $F = Gm_1m_2/r^2$ where r is the distance of separation. Initially the objects are a distance D apart. The work done in moving an object from position D to position x against a force F is defined as

$$W = \int_D^x F(r)\ dr.$$

Find the total work needed to move one of these objects infinitely far away.

12.11

"Gabriel's Horn" is the surface of revolution formed by rotating the graph of the function $y = f(x) = 1/x$ about the x axis for $1 \le x \le \infty$.

(a) Find the volume of air inside this shape and show that it is finite.

(b) When we cut a cross-section of this horn along the xy plane, we see a flat area which is wedged between the curves $y = 1/x$ and $y = -1/x$. Show that this "cross-sectional area" is infinite.

(c) The surface area of a surface of revolution generated by revolving the function $y = f(x)$ for $a \le x \le b$ about the x axis is given by

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\ dx$$

Write down an integral that would represent the surface area of "Gabriel's Horn".

(d) The integral in part (c) is not easy to evaluate explicitly - i.e. we cannot find an anti-derivative. However, we can show that it diverges. Set up a comparison that shows that the surface area of Gabriel's horn is infinite.

12.12

Suppose that x is a variable that takes on random non-negative values only.

(a) Show that the function below is a probability density distribution.

$$f(x) = \frac{2}{(1+x)^3},\ 0 \le x < \infty$$

Note: the value of x here is not bounded, so you will have to compute an integral whose limits are from 0 to infinity. Consider integrating from 0 to b and then letting b go to infinity.

(b) Find the probability that x takes on values in the interval $[1, 4]$

12.13

Given the probability distribution $p(x) = C\frac{1}{(1+x)^2}$, defined on $0 \le x < \infty$.

(a) Find the probability that x takes on values in the interval $[1, 4]$. (You will first have to find the value of the constant, C).

(b) Sketch both $p(x)$ and $F(x)$, the cumulative distribution function, for this probability density.

12.14

The probability that seeds will be dispersed a distance x (meters) away from a parent tree is found to be $p(x) = Ce^{-x/10}$.

(a) What is the mean dispersal distance?

(b) What is the variance in the dispersal distance?

(c) Seeds that fall right under the tree (for $x < 1$ meter) fail to thrive because of competition with the parent tree and shading that stunts the growth of the seedling. What fraction of the seeds will fail to thrive ? (Assume a one-dimensional arrangement.)

12.15

Consider the Normal (or Gaussian) distribution function given by

$$f(x) = Ce^{-\frac{x^2}{2}},\ -\infty < x < \infty,$$

and suppose you are told (since it is not easy to calculate) that

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}$$

(a) What value of the constant C should be used to make this function represent a probability density distribution? This is called the Standard Normal distribution.

(b) Show that the mean and median are both 0.

(c) Show that the variance and the standard deviation are both 1.

12.16

Consider the probability density distribution

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

(a) Show that $p(x)$ is a probability density function. Hint: use the substitution

$$z = \frac{x-\mu}{\sigma}.$$

(b) Show that the mean and median are both μ and the standard deviation is σ.

(c) Show that the most probable value of x occurs at $x = \mu$.

12.17

Quantum mechanics tells us that we can never know with complete certainty where a particle is at any particular time. Instead, it gives us a probability density which describes the chances that a particle is at some position. Einstein, as well as many others, found this difficult to accept, hence his famous quote "God does not play dice with the universe." Nevertheless, quantum mechanics has proven to be a theory which is remarkably consistent with physical observations.

For instance, if we are interested in how far the electron in a hydrogen atom is away from its nucleus, the probability density function is $p(r) = 4r^2e^{-2r}$, $(0 \leq r < \infty)$ where r is the distance measured in "Bohr radii". (One Bohr radius is the distance 5.29×10^{-11} meters.)

(a) Verify that $p(r)$ is a probability density function.

(b) Sketch the probability density function by determining $p(0), p'(r)$ and $\lim_{r\to\infty} p(r)$.

(c) Near which value of r is the electron most likely to be found?

(d) Find the mean distance.

(e) Find the cumulative distribution $F(r)$. What is the probability that the electron is found within 1 Bohr radius of the nucleus?

12.18

The Taylor series for the function $\sin(x)$ is given by

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots.$$

Find the Taylor series of

$$y = \cos(x)$$

by differentiating this function. Using the spreadsheet, plot (on the same graph) the following functions:

$$y = \cos(x), \quad y = T_1(x) \quad y = T_2(x), \quad y = T_3(x), \quad y = T_4(x)$$

where T_k is the polynomial made up of the first k (nonzero) terms in the Taylor series for $\cos(x)$.

12.19

An expansion for the function e^x is given by

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots = \sum_{n=0}^{\infty} \frac{x^k}{k!}.$$

Use the first seven terms of this series to estimate the value of the base of natural logarithms, i.e. of e^1.

12.20

Find the Taylor series for each of the following functions about $x = 0$:

(a) $f(x) = x\cos(x)$,

(b) e^{-x^2}

(c) $(1/x)\sin(x)$

(d) Use your results to find a Taylor series representation for each of the following integrals:

(i) $\int_0^x x\cos(x)dx$,

(ii) $\int_0^x e^{-x^2}dx$.

12.21

Find the Taylor series at $x = 0$ of the following functions:

(a) $f(x) = e^{-x^2/2}$.

(b) $f(x) = \ln(1 + x)$.

(c) $f(x) = \sqrt{x}\sin\sqrt{x}$.

(d) $f(x) = \dfrac{e^x + e^{-x}}{2}$.

12.22

Using the **Taylor series** of $f(x)$ near a point x^*, the value of a smooth function $f(x)$ in the vicinity of x^* can be approximated by $f(x^*)$, $f^{(k)}(x^*)$ (where $f^{(k)}(x)$ is the k^{th} derivative of $f(x)$), and the distance between x and x^*, $\Delta x = x - x^*$:

$$f(x) = f(x^*) + f'(x^*)\Delta x + \frac{f''(x^*)}{2!}(\Delta x)^2 + \frac{f'''(x^*)}{3!}(\Delta x)^3 + \cdots + \frac{f^{(k)}(x^*)}{k!}(\Delta x)^k + \cdots$$

where $k! = k \times (k-1) \times (k-2) \times \cdots \times 3 \times 2 \times 1$ is called the factorial of the integer k.

When $|\Delta x| << 1$ (i.e., when x is very close to x^*), $(\Delta x)^k$ can become negligibly small. If you only keep the first k terms of the series, the error in throwing away all the terms subsequent to the k^{th} term typically has a magnitude of the same order as $|\Delta x|^{k+1}$. This error can be made as small as you want by keeping more terms in the series. Thus, Taylor series is often used in calculating the approximate value of a function that is not a polynomial.

For example, knowing that $\sqrt{25} = 5$, we can calculate an approximate value of $\sqrt{26}$ by using the Taylor series. Note that,

$$\sqrt{26} = \sqrt{25+1} = \sqrt{25(1 + \frac{1}{25})} = 5\sqrt{1 + \frac{1}{25}}.$$

Now let $x^* = 1$ and $x = 1 + \frac{1}{25}$, thus $\Delta x = x - x^* = \frac{1}{25} = 0.04$ which is very small. Now using the Taylor series and keeping the first three terms:

$$\sqrt{x} \approx \sqrt{x^*} + [(\sqrt{x})']|_{x=x^*}\Delta x + \frac{[(\sqrt{x})'']|_{x=x^*}}{2!}(\Delta x)^2$$

which yields

$$\sqrt{1 + \frac{1}{25}} \approx \sqrt{1} + \frac{1}{2\sqrt{1}} \times 0.04 - \frac{1}{8\sqrt{1^3}} \times 0.04^2 = 1 + 0.02 - 0.0002 = 1.0198.$$

Thus,

$$\sqrt{26} = 5\sqrt{1 + \frac{1}{25}} \approx 5 \times 1.0198 = 5.099.$$

Using a calculator, we found that $\sqrt{26} = 5.0990195\ldots$ which is identical in the first 4 digits.

Use the first 3 terms of Taylor series to estimate the values of the following functions (Calculate by hand and compare the result with that obtained by a scientific calculator!)

(a) $\sqrt{110}$ (Hint: $110 = 100 + 10$ and $\sqrt{100} = 10$)

(b) $e^{0.1}$ (Hint: $0.1 = 0 + 0.1$ and $e^0 = 1$)

(c) $\cos(3)$ (Hint: $3 = \pi - (\pi - 3) \approx \pi - 0.14$ and $\cos(\pi) = -1$)

(d) $\tan^{-1}(1.1)$ (Hint: $\tan^{-1}(1) = \pi/4$, $(\tan^{-1}(x))' = 1/(1+x^2)$)

12.23

Find the Taylor series at $x = 0$, out to and including the x^n term, for the following functions:

(a) $f(x) = \tan x,\ n = 3.$

(b) $f(x) = (x+1)e^x,\ n = 4.$

(c) $f(x) = e^{x^2-2x},\ n = 3.$

(d) $f(x) = (\sin x)^2,\ n = 6.$

12.24

(a) Find a Taylor series for the function $f(x) = 1/(1+x^3)$ about $x = 0$. Show that this can be done by making the substitution $r = -x^3$ into the sum of a geometric series ($S = \sum r^k = \frac{1}{1-r}$).

(b) Use the same idea to find the Taylor series for the function $f(t) = 1/(1+t^2)$.

(c) Use your result in part (b) to find a Taylor series for the function $\tan^{-1}(x)$. (Hint: recall that $\tan^{-1}(x) = \int_0^x 1/(1+t^2)dt$).

12.25

Let $E(x)$ be the function defined by

$$E(x) = \int_0^x \frac{e^{-t} - 1}{t} dt.$$

(a) Use the Taylor series about $t = 0$ for the function e^{-t} to write down the analogous Taylor series about $t = 0$ for the function $\frac{e^{-t}-1}{t}$.

(b) Use your result in part (a) to determine the Taylor series about $x = 0$ for the integral $E(x)$.

12.26

Find a Taylor series about $x = 0$ for

$$F(x) = \int_0^x \ln(1 + t^2)dt.$$

12.27

Use a Taylor series representation to find the function $y(t)$ that satisfies the differential equation $y'(t) = y + bt$ with $y(0) = 1$. This type of equation is called a non-homogeneous differential equation. Show that when $b = 0$ your answer agrees with the known exponential solution of the equation $y'(t) = y$.

12.28

Use Taylor series to find the function that satisfies the following *second order* differential equation and initial conditions:

$$\frac{d^2y}{dt^2} + y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

(A second order differential equation is one in which a second derivative appears. Notice that this type of differential equation comes with *two* initial conditions.) Your answer should display $y(t)$ as a Taylor series expansion of the desired function. (You may be able to then guess what elementary function has this expansion as its Taylor series.)