Chapter 1

Areas, volumes and simple sums

1.1

Answer the following questions:

(a) What is the value of the fifth term of the sum $S = \sum_{k=1}^{20} (5+3k)/k$?

(b) How many terms are there in total in the sum $S = \sum_{k=7}^{17} e^k$?

(c) Write out the terms in $\sum_{n=1}^{5} 2^{n-1}$.

(d) Write out the terms in $\sum_{n=0}^{4} 2^n$.

(e) Write the series $1 + 3 + 3^2 + 3^3$ in summation notation in two equivalent forms.

1.2 Summation notation

(a) Write $2 + 4 + 6 + 8 + 10 + 12 + ...$ in summation notation.

(b) Write $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$ in summation notation.

(c) Write out the first few terms of $\displaystyle\sum_{i=0}^{100} 3^i$

(d) Write out the first few terms of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

(e) Simplify $\sum_{k=5}^{\infty}\left(\frac{1}{2}\right)^k + \sum_{k=2}^{4}\left(\frac{1}{2}\right)^k$

(f) Simplify $\sum_{x=0}^{50} 3^x - \sum_{x=10}^{50} 3^x$

(g) Simplify $\sum_{n=0}^{100} n + \sum_{n=0}^{100} n^2$

(h) Simplify $2\sum_{y=0}^{100} y + \sum_{y=0}^{100} y^2 + \sum_{y=0}^{100} 1$

1.3

Show that the following pairs of sequences are equivalent:

(a) $\sum_{m=0}^{10}(m+1)^2$ and $\sum_{n=1}^{11} n^2$

(b) $\sum_{n=1}^{4}(n^2-2n+1)$ and $\sum_{n=1}^{4}(n-1)^2$

1.4

Compute the following sums:

(a) $\sum_{i=1}^{290} 1$ (b) $\sum_{i=1}^{150} 2$ (c) $\sum_{i=1}^{80} 3$ (d) $\sum_{n=1}^{50} n$

(e) $\sum_{n=1}^{60} n$ (f) $\sum_{n=10}^{60} n$ (g) $\sum_{n=20}^{100} n$ (h) $\sum_{n=1}^{25} 3n^2$

(i) $\sum_{n=1}^{20} 2n^2$ (j) $\sum_{i=1}^{55} (i+2)$ (k) $\sum_{i=1}^{75} (i+1)$ (l) $\sum_{k=100}^{500} k$

(m) $\sum_{k=50}^{100} k$ (n) $\sum_{k=2}^{50} (k^2-2k+1)$ (o) $\sum_{k=5}^{50} (k^2-2k+1)$

(p) $\sum_{m=10}^{20} m^3$ (q) $\sum_{m=0}^{15} (m+1)^3$

For the solutions to these, we will use several summation formulae, and the notation shown below for convenience:

$$S_0(n) = \sum_{i=1}^{n} 1 = n \qquad S_1(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$S_2(n) = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad S_3(n) = \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

1.5

Use the sigma summation notation to set up the following problems, and then apply known formulae to compute the sums.

(a) Find the sum of the first 50 even numbers, $2 + 4 + 6 + ...$

(b) Find the sum of the first 50 odd numbers, $1 + 3 + ...$

(c) Find the sum of the first 50 integers of the form $n(n + 1)$ where $n = 1, 2, .., 50$.

(d) Consider all the integers that are of the form $n(n - 1)$ where $n = 1$, 2, $3 \ldots$. Find the sum of the first 50 such numbers.

1.6

Compute the following sum.

$S = \sum_{i=1}^{12} i(1 - i) + 2^i$

1.7

A clock at London's Heathrow airport chimes every half hour. At the beginning of the n'th hour, the clock chimes n times. (For example, at 8:00 AM the clock chimes 8 times, at 2:00 PM the clock chimes fourteen times, and at midnight the clock chimes 24 times.) The clock also chimes once at half-past every hour. Determine how many times in total the clock chimes in one full day. Use sigma notation to write the form of the series, and then find its sum.

1.8

A set of Japanese lacquer boxes have been made to fit one inside the other. All the boxes are cubical, and they have sides of lengths 1, 2, $3 \ldots 15$ inches. Find the total volume enclosed by all the boxes combined. Ignore the thickness of the walls of the boxes.

1.9

A framing shop uses a square piece of matt cardboard to create a set of square frames, one cut out from the other, with as little wasted as possible. The original piece of cardboard is 50 cm by 50 cm. Each of the "nested" square frames (see Problem 1.8 for the definition of nested) has a border 2 cm thick. How many frames in all can be made from this original piece of cardboard? What is the total area that can be enclosed by all these frames together?

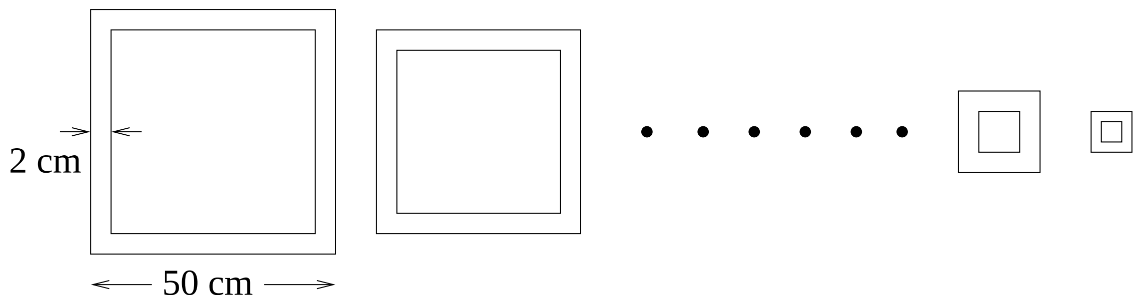

Figure 1.1: For problem 1.9

1.10

The Great Pyramid of Giza, Egypt, built around 2,720-2,560 BC by Khufu (also known as Cheops) has a square base. We will assume that the base has side length 200 meters. The pyramid is made out of blocks of stone whose size is roughly $1 \times 1 \times 0.73$ m^3. There are 200 layers of blocks, so that the height of the pyramid is $200 \cdot 0.73 = 146$ m. Assume that the size of the pyramid steps (i.e. the horizontal distance between the end of one step and the beginning of another) is 0.5 m. We will also assume that the pyramid is solid, i.e. we will neglect the (relatively small) spaces that make up passages and burial chambers inside the structure.

(a) How many blocks are there in the layer that makes up the base of the pyramid? How many blocks in the second layer ?

(b) How many blocks are there at the very top of the pyramid?

(c) Write down a summation formula for the total number of blocks in the pyramid and compute the total. (Hint: you may find it easiest to start the sum from the top layer and work your way down.)

1.11

Your local produce store has a special on oranges. Their display of fruit is a triangular pyramid with 100 layers, topped with a single orange (i.e. top layer : 1). The layer second from the top has three ($3 = 1 + 2$) oranges, and the one directly under it has six ($6 = 3 + 2 + 1$). The same pattern continues for all 100 layers. (This results in efficient "hexagonal" packing, with each orange sitting in a little depression created by three neighbors right under it.)

(a) How many oranges are there in the fourth and fifth layers from the top? How many in the Nth layer from the top ?

(b) If the "pyramid of oranges" only has 3 layers, how many oranges are used in total? What if the pyramid has 4, or 5 layers?

(c) Write down a formula for the sum of the total number of oranges that would be needed to make a pyramid with N layers. Simplify your result so that you can use the summation formulae for $\sum n$ and for $\sum n^2$ to determine the total number of oranges in such a pyramid.

(d) Determine how many oranges are needed for the pyramid with 100 layers.

1.12

A right circular cone (shaped like "an inverted ice cream cone"), has height h and a circular base (radius r) which is perpendicular to the cone's axis. In this exercise you will calculate the volume of this cone.

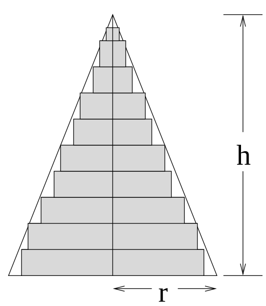

Figure 1.2: For problem 1.12, N=10

(1) Make N uniform slices of the cone, each one parallel to the bottom, and of height h/N. Inside each slice put a cylindrical disk of the same height. The radius of the slices vary from 0 at the top to nearly r at the bottom. (See Figure 1.2, $N = 10$.) Use similar triangles to answer these questions:

(a) What is the smallest disk radius other than 0?

(b) What is the radius of the kth disk?

(2) Express the total volume of the N disks as a sum.

(3) As N gets larger, what is the limit of this sum? (This is the volume of the cone.)

1.13

A (finite) geometric series with k terms is a series of the form $1 + r + r^2 + r^3 + ...r^{k-1} = \sum_{n=0}^{k-1} r^n$. The sum of such a series is $S = \frac{1-r^k}{1-r}$ provided $r \neq 1$.

(a) This formula does not work if $r = 1$. Find the actual value of the series for $r = 1$.

(b) Express in summation notation and find the sum of the series $1 + 2^1 + 2^2 + 2^3 + .. + 2^{10}$.

(c) Express in summation notation and find the sum of the series $1 + (0.5)^1 + (0.5)^2 + (0.5)^3 + ..(0.5)^{10}$.

1.14

Use the sum of a geometric series to answer this question.

(a) Find the sum of the first 11 numbers of the form $1.1^k, \quad k = 0, 1, 2, ...$ Now find the sum of the first 21 such numbers, the first 31 such numbers, the first 41 such numbers, and the first 51 such numbers.

(b) Repeat the process but now find sums of the numbers $0.9^k, \quad k = 0, 1, 2...$

(c) What do you notice about the pattern of results in part (a) and in part (b) ? Can you explain what is happening in each of these cases and why they are different?

(d) Now consider the general problem of finding a value for the sum

$$\sum_{k=0}^{N} r^k$$

when the number N gets larger and larger. Suggest under what circumstances this sum will stay finite, and what value that finite sum will approach. To do this, you should think about the formula for the finite geometric sum and determine how it behaves for various values of r as N gets very large. This idea will be very important when we discuss infinite series.

1.15

According to legend, the inventor of the game of chess (in Persia) was offered a prize for his clever invention. He requested payment in kind, i.e. in kernels of grain. He asked to be paid 1 kernel for the first square of the board, two for the second, four for the third, etc. Use a summation formula to determine the total number of kernels of grain he would have earned in total. (Hint: a chess board has $8 \times 8 = 64$ squares and the first square contains $2^0 = 1$ kernel.)

1.16

A branching colony of fungus starts as a single spore with a single segment of filament growing out of it. This will be called generation 0. The tip of the filament branches, producing two new segments. Each tip then branches again and the process repeats. Suppose there have been 10 such branching events. How many tips will there be? If each segment is the same length (1 unit), what will be the total length of all the segments combined after 10 branching events? (Include the length of the initial single segment in your answer.)

1.17 A branching plant and geometric series

A plant grows by branching, starting with one segment of length ℓ_0 (in the 0'th generation). Every parent branch has exactly two daughter branches. The length of each daughter branch is (2/5) times the length of the parent branch. (a) Find the total length of just the 12'th generation branch segments (b) Find the total length of the whole structure including the original segment and all 12 successive generations (c) Find the approximate total length of all segments in the whole structure if the plant keeps on branching forever. (Your answers will be in terms of ℓ_0.)

1.18 Branching airways, continued

Consider the branching airways in the lungs. Suppose that the initial bronchial segment has length ℓ_0 and radius r_0. Let α and β be the scale factors for the length and radius, respectively, of daughter branches (i.e. in a branching event, assume that $\ell_{n+1} = \alpha \ell_n$ and $r_{n+1} = \beta r_n$ are the relations that link daughters to parent branches). Let b be the average number of daughters per parent branch. Let $F_n = S_n/V_n$ be the ratio of total surface area to total volume in the n'th layer of the structure (i.e. for the n'th generation branches). Find F_n in terms of $\ell_0, r_0, b, \beta, \alpha$. In the lungs, it would be reasonable to expect that the surface area to volume ratio should *increase* from the initial segment down through the layers. What should be true of the parameters for this to be the case?

1.19 Dividing cells

In Figure 1.3, a cell labeled 0 (whose radius = 1mm) produces two daughter cells (layer 1) and these produce two daughters (layer 2) and so on. At each division, the radius of the daughters is 1/3 that of the parent cell.

(a) Set up a summation for the total cross-sectional area of these cells (i.e. areas of the circles) given that there are layers 0, 1, ... N with the same pattern. Then find the total area of the whole structure - i.e. the sum of the circular areas assuming that N is large. (Hint: Note that for large N, and $|a| < 1$, you can use the approximation $\sum_{k=0}^{N} a^k \approx 1/(1-a)$).

(b) Now find the total volume of this same collection of spherical cells.

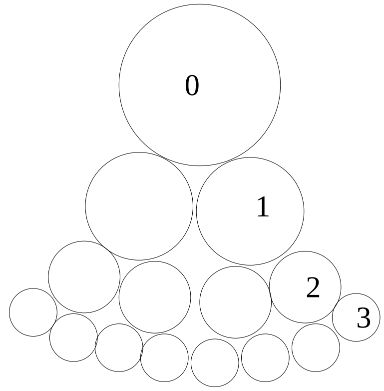

Figure 1.3: Figure for problem 1.19

1.20 Branching lungs

(a) Consider branched airways that have the following geometric properties (Table 1.20). Find the total number of branch segments, the volume and the surface area of this branched structure.

radius of first segment	r_0	0.5 cm
length of first segment	ℓ_0	5.0 cm
ratio of daughter to parent length	α	0.8
ratio of daughter to parent radius	β	0.8
number of branch generations	M	20
average number daughters per parent	b	2

(b) What happens as M gets larger? Will the volume and the surface area approach some finite limit, or will they grow indefinitely? How should the parameter β be changed so that the surface area will keep increasing while the volume stays finite as M increases?

1.21 Using simple geometry to compute an area 1

Some areas in the plane can be computed using simple geometric properties of triangles, circles, etc. Here is one such example.

A triangle is inscribed in a circle of radius $r = 1$, as shown in Figure 1.4. Triangle ABC is isosceles. Suppose that the angle BAC is 90°. Find the area of the triangle.

1.22 Using simple geometry to compute an area 2

(a) Find the area of a regular octagon (a polygon that has eight equal sides). Assume that the length of each side is 1 cm.

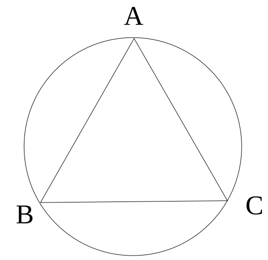

Figure 1.4: Figure for problem 1.21

(b) What is the area of the smallest circle that can be drawn around this octagon?

1.23 Approximation of π

The method used by Archimedes to compute the area of a circle can also be used to find an approximation for the value of π. Suppose we use a regular 36-sided inscribed polygon to approximate a circle of radius 1. Consider the thirty six triangles formed by dissecting the polygon into "pizza slices". You can find the area of one of these triangles if its height and base are known (and these can be found with a bit of trigonometry and geometry). Let the radius of the circle be one unit. (So what is its area using the now known formula for the area of a circle?). Find the area of the inscribed 36-sided polygon, and use your result to arrive at an approximation for the value of π.