2.1 Areas in the plane 1

(a) Compute the area of the staircase shown in Figure 2.1.

(b) What would be the area of that region if, instead of the ten steps shown, it consisted of 100 steps, each of width 0.1 and with heights 0.1, 0.2, ..10 ?

(c) If there are a very large number of steps of very small width, and very small height increments, what would be the approximate area of the region shaded?

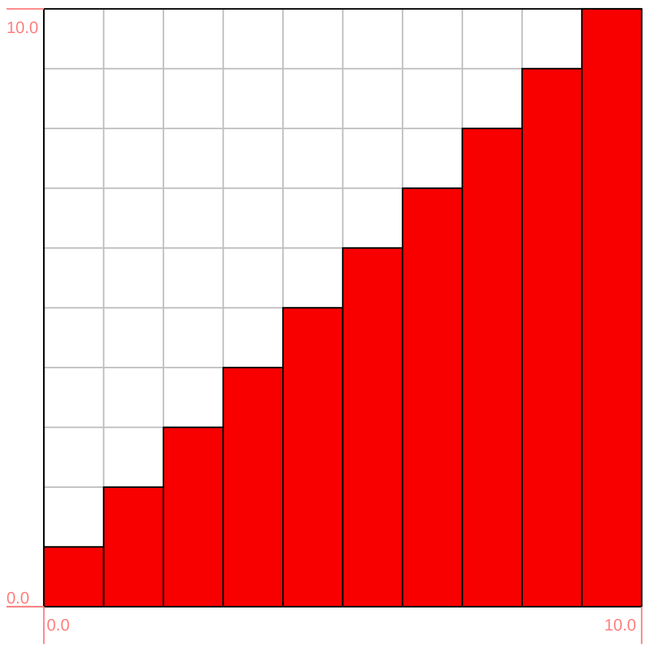


Figure 2.1:

2.2

Find the area bounded by the x-axis, the y-axis, and the graph of the function $y = f(x) = 1 - x$. (See Figure 2.2.)

(a) By using your knowledge about the area of a triangular region.

(b) By setting up the problem as a sum of the areas of N rectangular strips, using the appropriate summation formula, and letting the number of strips (N) get larger and larger to arrive at the result. Show that your answer in (b) is then identical to the answer in (a).

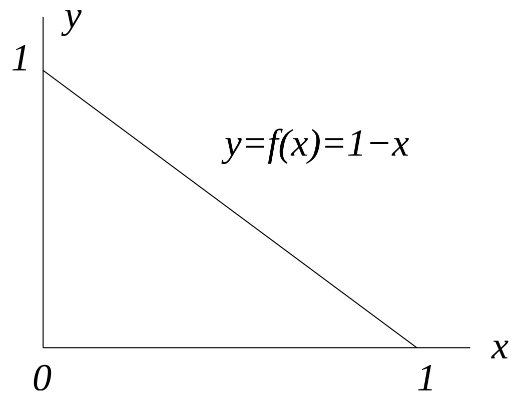


Figure 2.2: Figure for problem 2.2

2.3 Areas in the plane 2

Compute the areas of the two shaded regions for the interval $0 < x < 20$ in Figure 2.3. The curve shown on the diagram is $y = x^2$ and there are twenty rectangles forming the staircase. How do the areas of the shaded regions relate to the area under this curve?

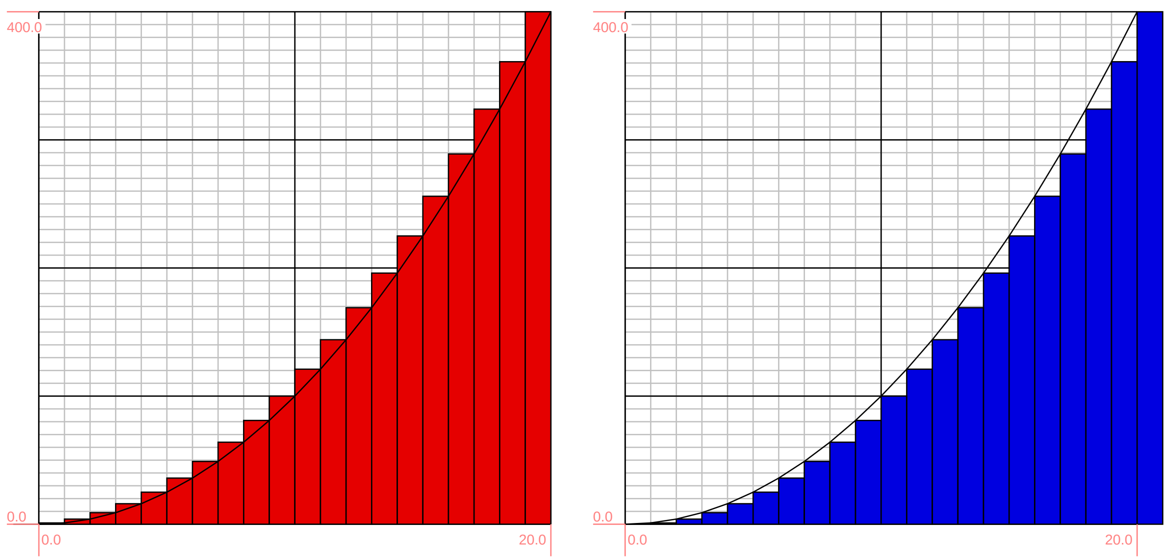


Figure 2.3:

2.4

Estimate the area under the graph of $f(x) = x^2 + 2$ from $x = -1$ to $x = 2$ in each of the following ways, and sketch the graph and the rectangles in each case.

(a) By using three rectangles and left endpoints.

(b) Improve your estimate in (a) by using 6 rectangles.

(c) Repeat part (a) using midpoints.

(d) Repeat part (b) using midpoints.

(e) From your sketches in parts (a), (b), (c) and (d), which appears to be the best estimate?

2.5

Find the area between the graphs of the functions: $y = f(x) = 2x$ and $y = g(x) = 1 + x^2$ between $x = 0$ and their intersection point.

2.6

Consider the function $y = f(x) = e^x$ on the interval $[0, 1]$.

Subdivide the interval $[0, 1]$ into 4 equal subintervals of width 0.25 and find an approximation to the area of this region using four rectangular strips.

(a) Use the left endpoints of each interval to obtain the heights of the rectangular strips.

(b) Use the right endpoints of each interval to obtain the heights of the rectangular strips.

(c) Explain the reason why your answer in (a) is different from your answer in (b).

2.7

Find the area between the x axis and the graph of the function $y = f(x) = 2 - x$ between $x = 0$ and $x = 2$.

(a) By using your knowledge about the area of a triangular region.

(b) By setting up the problem as a sum of the areas of n rectangular strips, using the appropriate summation formula, and letting the number of strips (n) get larger and larger to arrive at the result. Show that your answer in (b) is then identical to the answer in (a).

2.8

Find the area between the graphs of the following two functions: $y = f(x) = x^2$ and $y = g(x) = 2 - x^2$. (Hint: what do we mean by "between"? Where does this region begin and where does it end?) This problem should be set up in the form of a sum of areas of rectangular strips. You should NOT use previous familiarity with integration techniques to solve it.

2.9

Determine a function $f(x)$ and an interval on the x-axis such that the expression shown below is equal to the area under the graph of $f(x)$ over the given interval:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1+\frac{3i}{n}}$$

What do the terms that appear in this expression represent?

2.10

(a) Show that the area of a trapezoid with base b and heights h_1, h_2 (region shown on the left in Figure 2.4) is $A_{\text{trapezoid}} = \frac{1}{2}b(h_1 + h_2)$.

(b) Use the result for trapezoids to calculate the region under the graph of the function shown in Figure 2.4.

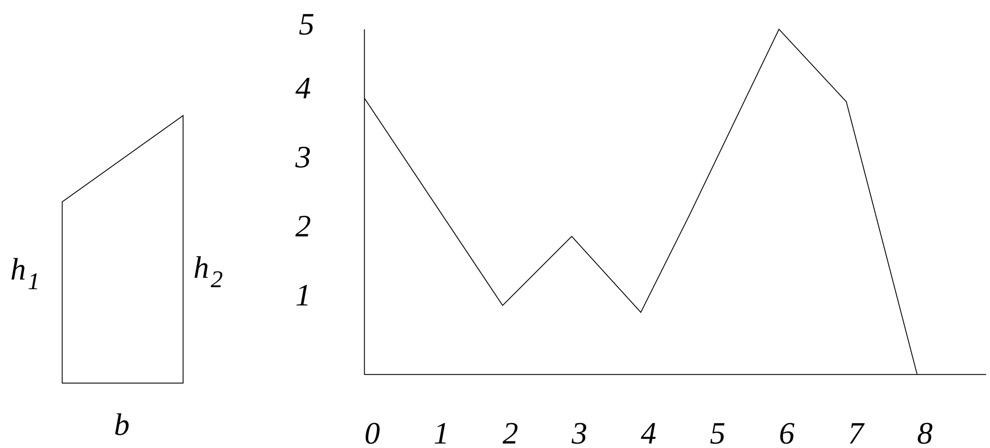


Figure 2.4: For problem 2.10

2.11

Consider the function $y = x^3$. Using the spreadsheet, create one plot which contains all of the following superimposed:

(a) a graph of this function for $0 < x < 1$.

(b) a bar-graph showing 20 rectangular strips, whose top *right* corner lies on the graph of the function.

(c) The function $A(x)$ which adds up the areas of the first strip, the first two strips, the first three strips.. etc. (See Figure 2.4 of the lecture notes for an example).

(d) The function $g(x) = x^4/4$ which is the anti-derivative of the original function.

2.12

Determine the values of each of the following definite integrals. Each of these can be done using simple geometry, and need no "integration" techniques. It will be helpful to sketch the regions and functions involved. Recall that we have formulae for areas of rectangles, triangles, and circles.

(a) $\int_0^1 2x dx$,

(b) $\int_{-1}^1 (1-x) dx$,

(c) $\int_{-2}^2 \sqrt{4-x^2} dx$

2.13

Consider the function $y = f(x) = x^3$. We would like to find the area under this curve for $0 < x < 3$ using the approximation by rectangular strips. Subdivide the interval into N regular subintervals such that $x_0 = 0, x_1 = \Delta x, \dots x_k = k\Delta x \dots x_N = 3$.

(a) What is the width of each interval (in terms of N)? Express x_k in terms of k and N.

(b) Consider N rectangles arranged so that the height of their top right corner is determined by the function $f(x)$. The first rectangle would have height $f(x_1)$, etc. Express the area of the k'th rectangle in terms of k and N.

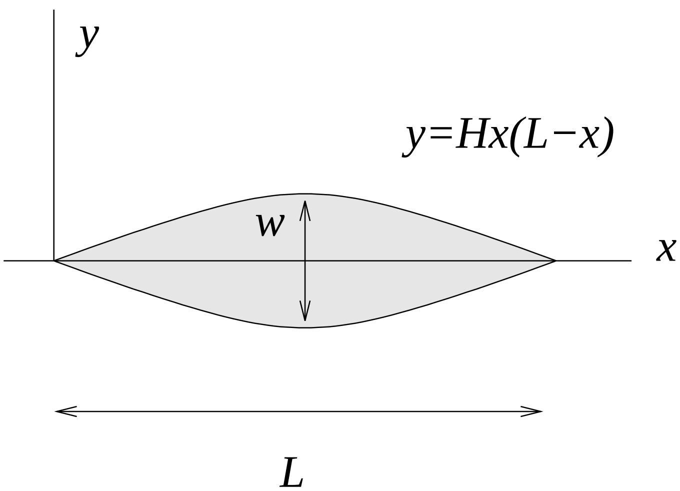
(c) Set up a sum of the areas of all these rectangles, and use the summation formula for the sum of cube integers to "add up" those areas and arrive at a total area A_N associated with those N rectangular strips. Your answer should be expressed in terms of N.

(d) Now consider what happens to A_N when the number of rectangles, N gets large. Find the value of the area A under the curve by taking a limit as $N \to \infty$.

2.14

In problem 2.13, we found the area under the function $f(x) = x^3$ for the interval $0 < x < 3$. Use your results from that problem to now determine the area under the same function over the interval $2 < x < 3$. (Hint: rather than redoing the entire calculation, think of how you could find this area by subtraction of two areas that start at $x = 0$.)

2.15 Leaf shape



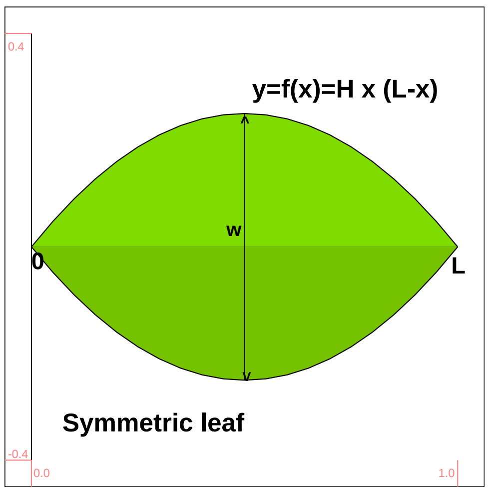


Figure 2.5: The shape of a symmetric leaf of length L and width w is approximated by the quadratic $y = Hx(L - x)$ in problem ??.

(a) The function $y = f(x) = Hx(L - x)$, shown in Figure 2.5 could approximately describe the shape of the (top edge) of a symmetric leaf of length L and width w for a particular choice of the constant H (in terms of w and L). Find the appropriate value of H. Assume that the width is the distance between the leaf edges at the midpoint of the leaf.

(b) Find the area between the x axis and the function $y = f(x) = Hx(L - x)$.

(c) Use your result from (a) to express the area of the leaf in terms of the width and length of this leaf.

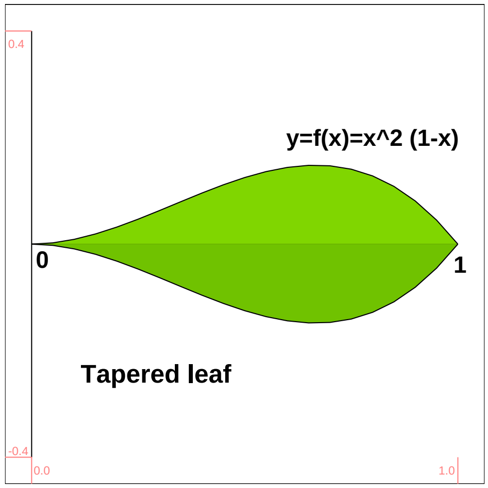


Figure 2.6: The shape of a tapered is approximated by the cubic $y = x^2(1-x)$.

2.16 Tapered Leaf

(a) Consider the shape of a leaf shown in Figure 2.6, and given by the function $y = f(x) = x^2(1-x)$. This leaf is not fully symmetric, since it is tapered at one end. By choice of the function that describes its top edge, the length of the leaf is 1 unit. Find the width of the leaf (distance between edges at the widest place). [Hint: use differential calculus to determine where the widest point occurs.]

(b) Find the area of this shape by dissection into rectangles.