Chapter 3

The Fundamental Theorem of Calculus

3.1

(a) Give a concise statement of the Fundamental Theorem of Calculus.

(b) Why is it a useful practical tool?

3.2

Consider the function $y = f(x) = e^x$ on the interval $[0, 1]$.

Find the area under the graph of this function over this interval using the Fundamental Theorem of Calculus.

3.3

Determine the values of the integrals shown below, using the Fundamental Theorem of Calculus (i.e. find the anti-derivative of each of the functions and evaluate at the two endpoints.)

(a) $\int_0^1 2x dx$

(b) $\int_{-1}^1 (1 - x) dx$

3.4

Use the Fundamental Theorem of Calculus to compute each of the following integrals. The last few are a little more challenging, and will require special care. Some of these integrals may not exist.

Explain why.

(a) $\int_0^{\pi} \sin(x)\,dx$ (b) $\int_0^{\frac{\pi}{4}} 2\sin(x)\,dx$ (c) $\int_0^{\frac{\pi}{2}} \cos(x)\,dx$

(d) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\cos(x)\,dx$ (e) $\int_2^{3} (x-2)\,dx$ (f) $\int_{-1}^{1} (x-1)\,dx$

(g) $\int_{-1}^{1} (x^2+1)\,dx$ (h) $\int_0^{4} (x+1)^2\,dx$ (i) $\int_0^{4} x^{1/2}\,dx$

(j) $\int_0^{1} 3\,x^{1/2}\,dx$ (k) $\int_1^{4} (1+\sqrt{x})\,dx$ (l) $\int_1^{2} (3/x)\,dx$

(m) $\int_1^{3} (2/x)\,dx$ (n) $\int_0^{1} 2e^x\,dx$ (o) $\int_{-2}^{-3} x^{1/3}\,dx$

(p) $\int_{-1}^{0} x^{1/2}\,dx$ (q) $\int_{-1}^{1} x^{-2}\,dx$ (r) $\int_{-1}^{1} 2|x|\,dx$

(s) $\int_0^{2} (1/x)\,dx$ (t) $\int_{-1}^{1} (2/x)\,dx$

3.5

Find the following integrals using the Fundamental theorem of Calculus.

(a) $\int_a^{x} e^{kt}\,dt$ (b) $\int_0^{x} A\cos(ks)\,ds$ (c) $\int_b^{x} Ct^m\,dt$ (d) $\int_0^{x} \frac{1}{aq}\,dq$

(e) $\int_c^{T} sec^2(5x)\,dx$ (f) $\int_1^{x} \frac{2}{1+t^2}\,dt$ (g) $\int_b^{x} \frac{3}{s^2}\,ds$

(h) $\int_a^{T} \frac{1}{x^{1/2}}\,dx$ (i) $\int_0^{x} (sin(3y))\,dy$ (j) $\int_b^{x} 3\,dt$

3.6

(a) Use an integral to estimate the sums $\sum_{k=1}^{N} \sqrt{k}$.

(b) For $N = 4$ draw a sketch in which the value computed by summing the four terms is compared to the value found by the integration. (Your sketch should show the graph of the appropriate function and a set of steps that represent the above sum.)

3.7

Find the area under the graphs of these functions:

(a) $f(x) = 1/x$ between $x = 1$ and $x = 3$.

(b) $v(t) = at$ between $t = 0$ and $t = T$, where a, T are fixed constants and $a > 0$.

(c) $h(u) = u^3$ between $u = 1/2$ and $u = 2$.

3.8

Find the area between the two curves $y = 1 - x$ and $y = x^2 - 1$ for $x > 0$. Explain the relationship of your answer to the two integrals $I_1 = \int_0^1 (1 - x)dx$ and $I_2 = \int_0^1 (x^2 - 1)dx$.

3.9

Find the area between the graphs of $y = f(x) = 2 - 3x^2$ and $y = -x^2$.

3.10

(a) Find the area enclosed between the graphs of the functions $y = f(x) = x^n$ and the straight line $y = x$ in the first quadrant. (Note that we are considering positive values of n and that for $n = 1$ the area is zero.)

(b) Use your answer in part (a) to find the area between the graphs of the functions $y = f(x) = x^n$, $y = g(x) = x^{1/n}$ in the first quadrant. (Hint: what is the relationship between these two functions and what sort of symmetry do their graphs satisfy?)

3.11

A piece of tin shaped like a leaf blade is to be cut from a square sheet of tin. The shape of one of the sides is given by the function $y = x^2$ and the shape is to be symmetric about the line $y = x$. How much tin goes into making the shape? How much is left over? Assume that the thickness of the sheet is such that each square cm weighs one gm, and that y and x are in meters.

3.12 1/3

Find the area between the graphs of the functions $y = f(x) = 2x$, $y = g(x) = 1 + x^2$ and the y axis.

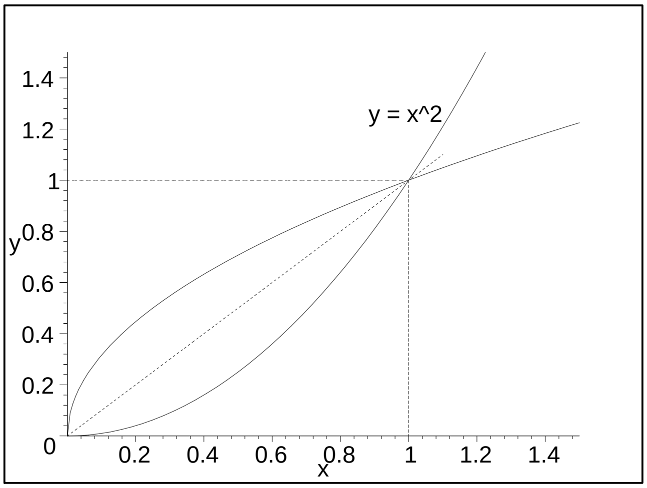

Figure 3.1: For problem 3.11

3.13

Find the area of the finite plane region bounded by the parabola $y = 6 - x^2$ and $y = x^2 - 4x$. [**from the April 97 Final Exam**].

3.14

Find the area of the shape shown in Figure 3.2.

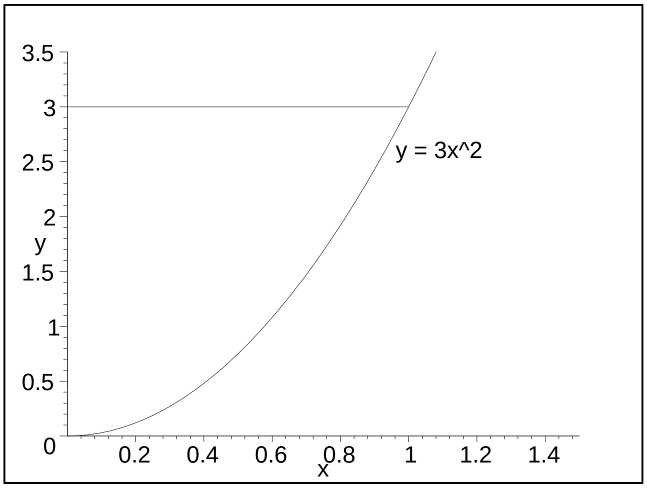

Figure 3.2: For problem 3.14

3.15

Find the area under the function shown below between $x = 1$ and $x = 5$. To do so, figure out the equations of the lines and curves making up this figure and use integration methods. The last part of the curve is parabolic.

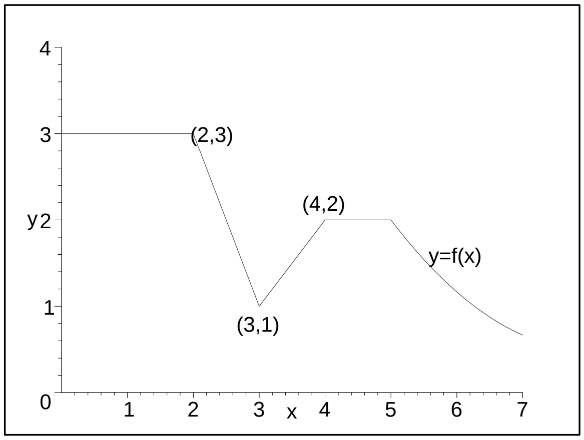

Figure 3.3: For problem 3.15

3.16

Let $g(x) = \int_0^x f(t)dt$, where $f(t)$ is of the function whose graph is shown.

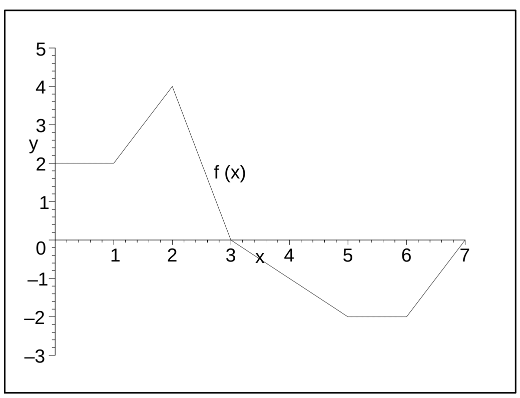

Figure 3.4: For problem 3.16

(a) Evaluate $g(0)$, $g(1)$, $g(2)$, $g(3)$ and $g(6)$.

(b) On what intervals is $g(x)$ increasing?

(c) Where does $g(x)$ have a maximum value?

(d) Sketch a rough graph of $g(x)$.

3.17

Consider the functions shown in Figure 3.5(a) and (b). In each case, use the sketch of this function $y = f(x)$ to draw a sketch of the graph of the related function $F(x) = \int_0^x f(t)dt$. (Assume that $F(-1) = -4$ in (a), and $F(x) = -0.5$ at the left end of the interval in (b).)

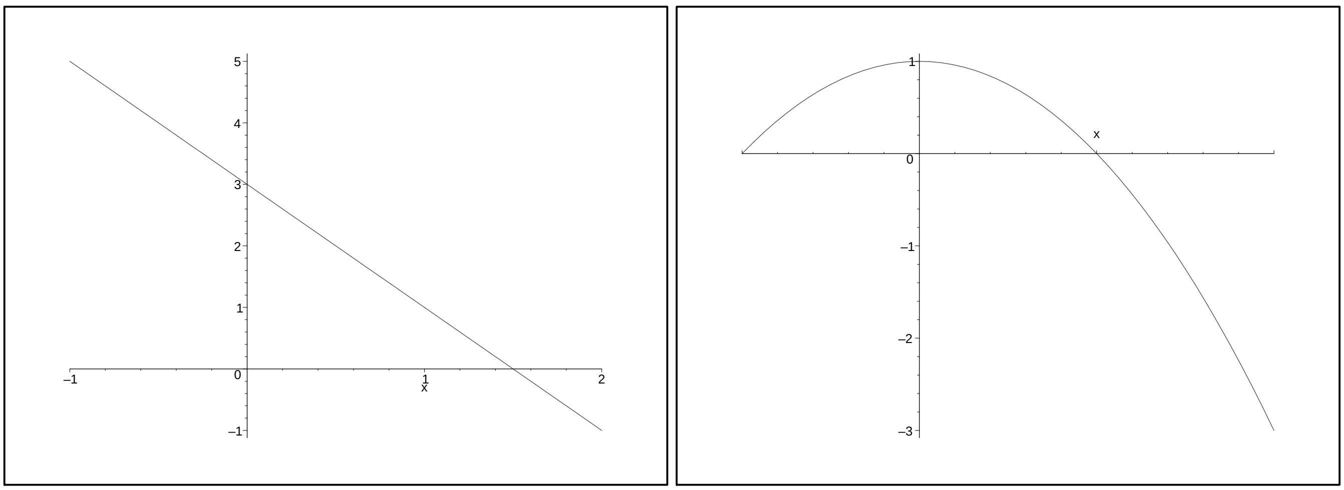

Figure 3.5: For problem 3.17

3.18

Consider the functions shown in Figure 3.6(a) and (b). In each case, use the sketch of this function $y = f(x)$ to draw a sketch of the graph of the related function $F(x) = \int_0^x f(t)dt$.

3.19

Consider the functions shown in Figure 3.7(a) and (b). In each case, use the sketch of this function $y = f(x)$ to draw a sketch of the graph of the related function $F(x) = \int_0^x f(t)dt$. (Assume that $F(0) = 0$ in both (a) and (b).)

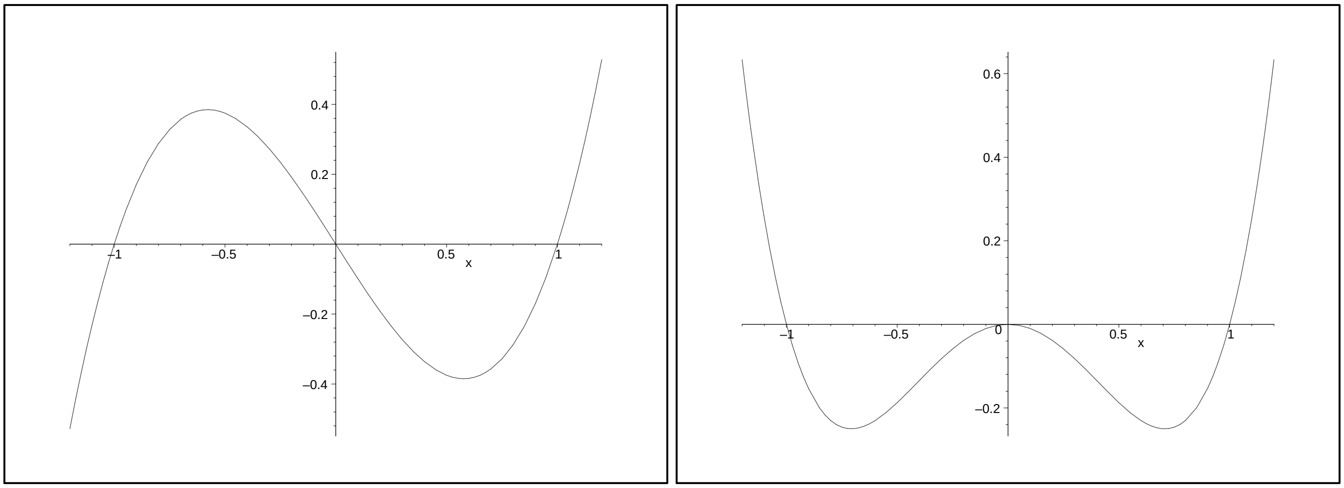

Figure 3.6: For problem 3.18

3.20 Leaves revisited

Use the Fundamental Theorem of Calculus (i.e. integration techniques) to find the areas of the leaves shown in Figure 3.8. These leaves are generated by the functions

(a) $y = x^2(1 - x)$,

(b) $y = x(1 - x)^2$,

(c) $y = x(1 - x)$,

(d) $y = x(1 - x^2)$.

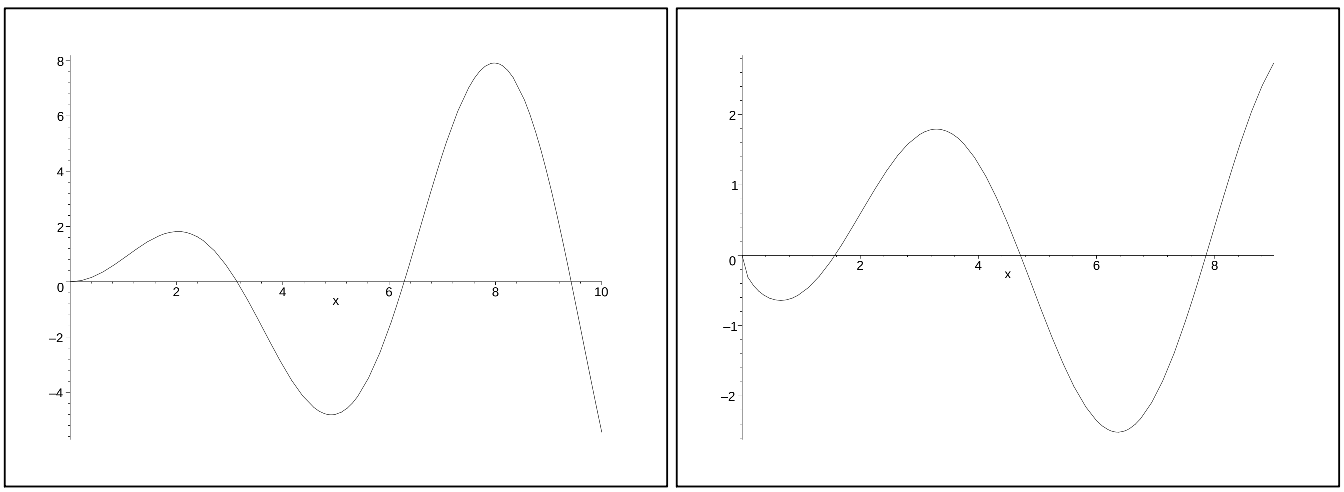

Figure 3.7: For problem 3.19

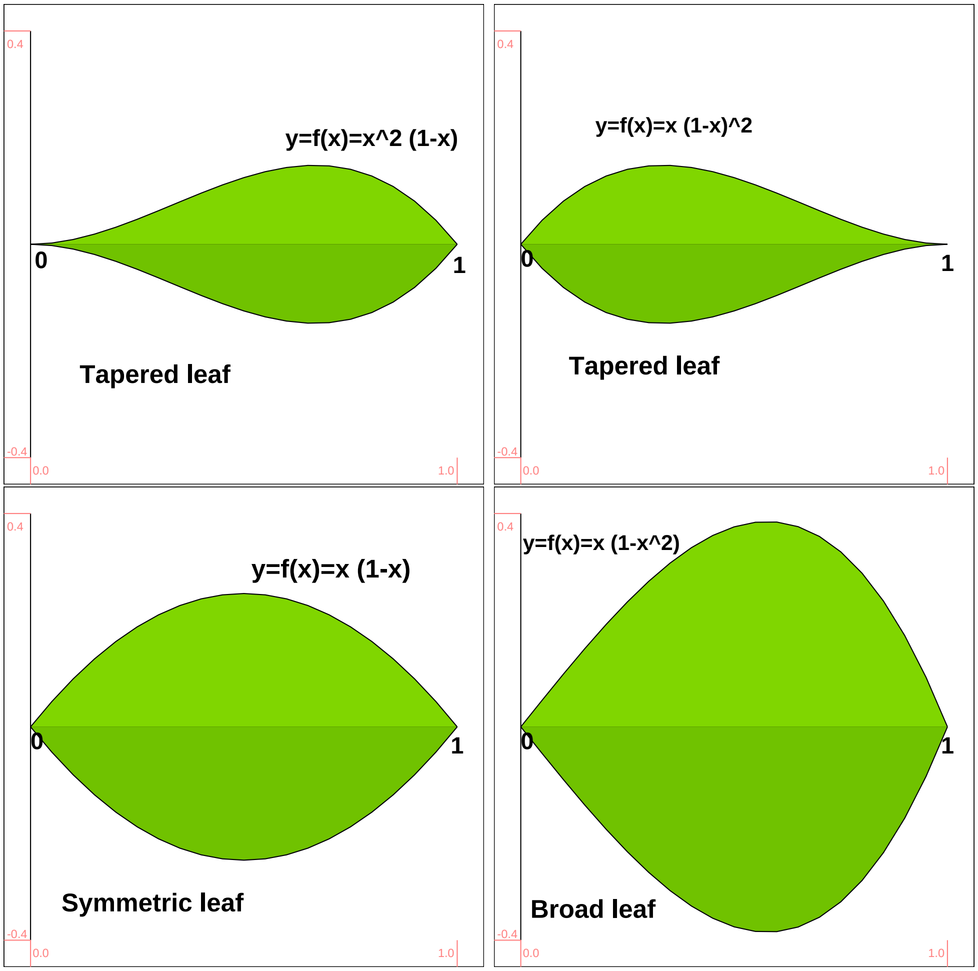

Figure 3.8: The shape of some leaves in problem 3.20.