Chapter 6

Techniques of Integration

6.1 Differential Notation 1

Calculate the differential of the following functions by using the definition

$$dy = y'(x)dx.$$

Express the result in terms of the product between $y'(x)$ and the differential of x, dx.

For example, given

$$y(x) = 3x + \sin(2x),$$

its differential is

$$dy = y'(x)dx = (3 + 2\cos(2x))dx.$$

(a) $f(x) = e^{x^2}$	(b) $f(x) = (x + 1)^2$	(c) $f(x) = \sqrt{x}$
(d) $f(x) = \arcsin(x)$	(e) $f(x) = x^2 + 3x + 1$	(f) $f(x) = \cos(2x)$
(g) $y = x^6 + 2x^4 - 2x$	(h) $y = (x - 2)^2(x + 1)^5$	(i) $y = x/(x + 3)$

6.2 Differential Notation 2

Given the differential of a function y in terms of the product between its derivative $y'(x)$ and the differential of x, (dx), express the same differential in terms of the the differential of y itself, i.e., dy. Then, use the result combined with the fundamental theorem of calculus to calculate the corresponding indefinite integrals.

For example, if we know that the differential of a function $y(x)$ is given by

$$(3 + 2\sin(2x))dx =?, \quad \text{(Or equivalently, } \int (3 + 2\sin(2x))dx =?)$$

we now need to figure out that $y(x) = 3x - \cos(2x)$ and express the differential given above in terms of the differential of y itself

$$(3 + 2\sin(2x))dx = d(3x - \cos(2x)).$$

Using this result and the fundamental theorem of calculus, we can solve the following

$$\int (3 + 2\sin(2x))dx = \int d(3x - \cos(2x)) = 3x - \cos(2x) + C.$$

(Those marked by a star are a little bit more difficult since they involve composite functions.)

(a) $x^3 dx =?$, $\int x^3 dx =?$ (b) $x^{3/2} dx =?$, $\int x^{3/2} dx =?$
(c) $\frac{1}{x^3} dx =?$, $\int \frac{1}{x^3} dx =?$ (d) $\sqrt{x} dx =?$, $\int \sqrt{x} dx =?$
(e) $\frac{1}{\sqrt{x}} dx =?$, $\int \frac{1}{\sqrt{x}} dx =?$ (f) $e^{-2x} dx =?$, $\int e^{-2x} dx =?$
(g) $(x+1)^2 dx =?$, $\int (x+1)^2 dx =?$ (h) $\frac{1}{x+3)^2} dx =?$, $\int \frac{1}{x+3)^2} dx =?$
(i) $\cos(2x) dx =?$, $\int \cos(2x) dx =?$ (j) $\sec^2(x) dx =?$, $\int \sec^2(x) dx =?$
(k) $\frac{1}{x} dx =?$, $\int \frac{1}{x} dx =?$ (l) $\frac{1}{1+x^2} dx =?$, $\int \frac{1}{1+x^2} dx =?$
(m)* $2xe^{x^2} dx =?$, $\int 2xe^{x^2} dx =?$ (n)* $3x^2 \cos(x^3) dx =?$, $\int 3x^2 \cos(x^3) dx =?$
(o)* $\frac{2x}{1+x^2} dx =?$, $\int \frac{2x}{1+x^2} dx =?$ (p)* $\frac{2x}{1+x^4} dx =?$, $\int \frac{2x}{1+x^4} dx =?$

6.3 Substitution 1

(a) Evaluate the following indefinite integrals using the substitution method:

$$\int \sin(3x)\ dx, \quad \int \frac{\cos(\sqrt{x})}{\sqrt{x}}\ dx, \quad \int x^3\sqrt{x^4+1}\ dx, \quad \int \frac{4}{1+2x}\ dx.$$

(b) Calculate the following definite integrals using the substitution rule:

$$\int_0^5 (\sqrt{3+2x})\ dx, \quad \int_0^{\pi/4} \sin(4t)\ dt, \quad \int_0^{\sqrt{\pi}} x\cos(x^2)\ dx.$$

6.4 Substitution 2

Compute the following integrals using substitution.

(a) $\int_1^p \frac{1}{1+y^2}\ dy$ (b) $\int x^2(x^3+1)^6\ dx$ (c) $\int e^x\sqrt{1+2e^x}\ dx$

(d) $\int \frac{1}{x\ln(x)}\ dx$ (e) $\int \frac{1}{1-y}\ dy$ (f) $\int_1^S \frac{k_1}{k_2-n}\ dn, \quad k_1, k_2 > 0$

(g) $\int \frac{x^2}{1-x^3}\ dx$ (h) $\int \sqrt{3x+1}\ dx$ (i) $\int \frac{x}{\sqrt{4+x^2}}\ dx$

(j) $\int \cos(x)\sin^5(x)\ dx$ (k) $\int \frac{3}{4+5x}\ dx$ (l) $\int \cot(\theta)\ d\theta$

(m) $\int \frac{\sec^2(x)}{\sqrt{2+\tan(x)}}\ dx$ (n) $\int \frac{2}{4+x^2}\ dx$

6.5 Trigonometric Substitution 1

The integral $\int \sin(x)\cos(x)dx$ can be done in several ways:

(a) using the substitution $u = \sin(x)$ (or $u = \cos(x)$) OR

(b) by first using the trigonometric identity $\sin(2x) = 2\sin(x)\cos(x)$ and then integrating. Show that the two answers are equivalent. (Hint: you will find that this is a good opportunity to review trigonometric identities.)

6.6 Trigonometric substitution 2

The integral $\int_0^1 \sqrt{1-x^2}\,dx$ can be done by making the substitution $x = \sin(u)$ and $dx = \cos(u)du$. This is called a trigonometric substitution.

(a) Show that this reduces the integral to $\int \cos^2(u)\,du$.

(b) Now use the identity $\cos^2(u) = \dfrac{1+\cos(2u)}{2}$ to re-express the integral in a simpler form. Then integrate.

(c) Explain why the answer is the same as the area of 1/4 of a circle of radius 1.

6.7 Partial Fractions 1

Practice with Integration: Compute the following integrals. Use factoring and/or completing the square and partial fractions, or some other technique if necessary.

(a)

$$\int \frac{1}{x^2 - x - 20}\,dx$$

(b)

$$\int \frac{3}{x^2 + 6x + 9}\,dx$$

(c)

$$\int \frac{-1}{x^2 + 4x + 14}\,dx$$

(d)

$$\int \frac{2}{x^2 - 6x + 8}\,dx$$

6.8 Partial Fraction 2

The integrals shown below may look very similar, but in fact they lead to quite different results:

(a)

$$\int \frac{dx}{a^2 + x^2}$$

(b)

$$\int \frac{dx}{a^2 - x^2}$$

Show that (a) can be reduced to an inverse tangent type integral by a bit of algebraic rearrangement and a substitution of the form $u = x/a$. Show that (b) can be integrated by factoring the expression $a^2 - x^2$ and using the method of partial fractions.

6.9 Integration by Parts 1

Use integration by parts to show that $\displaystyle\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$

6.10 Integration by Parts 2

First make a substitution and then use integration by parts to evaluate the following integrals:

(a)

$$\int \sin(\sqrt{x})\ dx$$

(b)

$$\int_1^4 e^{\sqrt{x}}\ dx$$

(c)

$$\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} x^3 \cos(x^2)\ dx$$

(d)

$$\int x^5 e^{x^2}\ dx$$

6.11

Compute the following integrals.

(a) $\displaystyle\int \frac{3}{x^2+4x+6}$

(b) $\displaystyle\int \frac{2}{(x+1)(x+2)}\,dx$ (c) $\displaystyle\int \frac{1}{x^2+6x+8}\,dx$ (d) $\displaystyle\int_1^p \frac{1}{1-y^2}\,dy$

(e) $\displaystyle\int_0^T te^{-2t}\,dt$ (f) $\displaystyle\int_0^\pi x\sin(\frac{x}{2})\,dx$ (g) $\displaystyle\int x^2\ln(x)\,dx$

6.12

Integration drills: (Note: some simplifications in a few of these will make your work much easier.)

(1) $\displaystyle\int \frac{1}{x}\ln(x)\,dx$ (2) $\displaystyle\int \frac{2}{2+3x}\,dx$ (3) $\displaystyle\int x^2\sin(3x^3+1)\,dx$

(4) $\displaystyle\int_1^2 2t\,e^{3t^2}\,dt$ (5) $\displaystyle\int_0^1 \frac{x}{1+x^2}\,dx$ (6) $\displaystyle\int \frac{1}{\sqrt{2-3x^2}}\,dx$

(7) $\displaystyle\int \frac{2}{3+4x^2}\,dx$ (8) $\displaystyle\int \frac{1}{\sqrt{25-4x^2}}\,dx$ (9) $\displaystyle\int x\frac{1}{\sqrt{9+x^2}}\,dx$

(10) $\displaystyle\int \cos(3x)\,(1-\sin(3x))\,dx$ (11) $\displaystyle\int \cos(2x)(1-\sin(2x))\,dx$

(12) $\displaystyle\int \sec^2(x)\,\sqrt{\tan(x)+1}\,dx$ (13) $\displaystyle\int_0^{\pi/2} (1-\sin^2(t))\,\sin(t)\,dt$

(14) $\displaystyle\int 2x(1-\sin(2x))\,dx$ (15) $\displaystyle\int \sec(x)\tan(x)\,dx$ (*Hint: use your trig relations*)

(16) $\displaystyle\int x\,\sin(x+1)\,dx$ (17) $\displaystyle\int (x\sin^2(x)+x\cos^2(x))^5\,dx$ (*Hint: look carefully!*)

(18) $\displaystyle\int \sec^2(t)\,dt$ (19) $\displaystyle\int \frac{1}{2x^2+12x+18}\,dx$ (*Some algebra, please*)

(20) $\displaystyle\int \frac{1}{(x-5)(x+1)}\,dx$ (21) $\displaystyle\int \frac{1}{(x+3)(2x-1)}\,dx$

(22) $\displaystyle\int \frac{1}{(x+2)(x+3)}\,dx$ (23) $\displaystyle\int \frac{3x^2+1}{x^2(x^2+1)^2}\,dx$

(24) $\displaystyle\int \frac{1}{x^2-3x+2}\,dx$ (*Hint: try factoring*) (25) $\displaystyle\int \frac{3}{x^2+3x+15}\,dx$

(26) $\int x\sec^2(x)\ dx$ (27) $\int xe^{x^2}\ dx$ (28) $\int x^2e^x\ dx$

(29) $\int \tan^{-1}(x)\ \ dx$ (30) $\int\ x\ \cos(x)\ e^x\ dx$