Chapter 7

Distributions and center of mass

7.1

In a class of 20 students writing a test worth 10 points, 5 students scored 6 points, 5 scored 8 points, and 10 scored 9 points. Find the average score achieved by this class on the test.

7.2

Five beads are distributed along a thin 1 dimensional wire. Their masses and positions are: $m_1 = 5, x_1 = 0; m_2 = 2, x_2 = 3; m_3 = 1, x_3 = 4; m_4 = 2, x_4 = 5; m_5 = 10, x_5 = 6$. Find the total mass and the center of mass of this discrete mass distribution.

7.3

Suppose that the function $\rho(x)$ represents the density of a bar for $a \leq x \leq b$,. Explain the distinction between:

(a) the average density of the bar, $\bar{\rho}$

(b) the mass of the bar, M, and

(c) the center of mass (or centroid) of the bar, $\bar{x}$.

7.4

The density of a bar is given by

$$\rho(x) = 1 - x \quad 0 < x < 1$$

(a) Sketch the density of the bar and the function

$$M(x) = \int_0^x p(s)\ ds.$$

Find the total mass of the bar.

(b) Find the average mass density along the bar.

(c) Find the center of mass of the bar.

(d) Where along the length of the bar should you cut to get two pieces of equal mass?

(e) What fraction of the mass of the bar is found between $x = 0$ and $x = 0.5$?

7.5

The density of a bar is given by

$$\rho(x) = ax^2 \quad 0 < x < L$$

(a) Find the total mass of the bar.

(b) Find the average mass density along the bar.

(c) Find the center of mass of the bar.

(d) Where along the length of the bar should you cut to get two pieces of equal mass?

7.6

Let the density of mass along a bar of length L be given by

$$p(x) = e^{-ax}$$

(a) Find the total mass of the bar.

(b) Find the average mass density along the bar.

(c) Find the center of mass of the bar.

7.7

Find the center of mass of a distribution

$$p(x) = \sin(2x) \quad 0 < x < \frac{\pi}{2}$$

7.8

The density of a beam is given by the function $\rho(x) = x^{m/n}$ where $0 \leq x \leq 1$.

(a) Find the center of mass $\bar{x}$.

(b) Explain what happens to the center of mass if m is very large (for $n = 1$).

(c) What happens to the center of mass if n is very large, (for $m = 1$)?

7.9

To investigate changes in the Earth's weather, scientists examine the distribution of pollen grains in a 1 dimensional drilled "core sample", i.e. a sample of the Earth's crust that contains archaeological deposits of soil from many thousands of years. Suppose that pollen grain distribution in a core sample of length L is given by

$$p(x) = A\sin(ax), \qquad 0 < x < L = \frac{\pi}{a}.$$

where $p(x)$ are the number of particles per unit volume at a distance x from one end of the sample.

(a) Where is the pollen grain most concentrated along this one dimensional sample?

(b) Find the average density of pollen grains along the length of the sample.

(c) Find the center of mass of the pollen grain distribution.

7.10

Bacteria are grown in a 1 dimensional tube. The mass density of the bacteria per unit distance along the tube is given by the function

$$b(x) = \beta\sqrt{a^2 - x^2}, \quad 0 < x < a.$$

(a) Find the total mass of bacteria in the tube.

(b) Find the center of mass of the bacteria.

7.11

Gel electrophoresis is an experimental technique used in molecular biology to separate proteins (or other molecules) according to their molecular weights and charges. Suppose that in such an

experiment, the distribution of protein along a 1 dimensional strip of this gel is found to be $p(x) = xe^{-x/2}$ where $0 \leq x \leq 5$ is distance in cm from the end of the strip and $p(x)$, is the density of the protein per unit distance (in arbitrary units).

(a) Sketch a graph of this distribution.

(b) Determine the location $x = x_c$ where the density of the protein is greatest. (Indicate on the graph and find using calculus.)

(c) Find the total amount of protein in the region $x_c - 1 \leq x \leq x_c + 1$.

(d) Find the mean (i.e. center of mass) of the distribution, i.e. the average x coordinate of the protein distribution.

7.12 Centroid 1

Find the x coordinate of the centroid of the following shape:

(a) The triangle with vertices at $(0, 0), (a, 0), (b, c)$.

(b) The quadrilateral with vertices at $(0, 0), (a, 0), (b, c), (b + a, c)$.

7.13 Centroid 2

Find the centroid of the semi-elliptical region defined by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \ x \geq 0.$$