Chapter 8

Discrete probability and the laws of chance

8.1 Multiple Events and Combined Probabilities 1

Determine the probability of each of the following events assuming that the die has equal probability of landing on each one of the six sides marked by 1 to 6 dots and that the coin has equal probability to land on head (H) and tail (T).

(a) The probability of rolling a 1, followed by a 4, in two consecutive rolls.

(b) The probability of NOT rolling either a 2 or a 3 in one roll.

(c) The probability of tossing a coin 3 times and getting TTH.

(d) The probability of getting any combination other than TT in two consecutive rolls.

(e) The probability of getting a coin landing on H and a die landing on 6 with one toss of coin and one roll of dice.

8.2 Multiple Events and Combined Probabilities 2

(a) Find the probability of randomly selecting 4 aces from a well-shuffled deck of 52 cards.

(b) Find the probability of randomly selecting 4 hearts (of any value) from a well-shuffled deck of 52 cards. (Note: a full deck of cards contains 13 hearts.)

(c) What is the probability of randomly selecting the sequence of cards of value "King", "Queen" Jack" (of any suit or combination of suits) from a well-shuffled deck of 52 cards.

8.3 Two coin tosses

List all the possible events in a 2-toss coin experiment, and assign these probabilities using the rules of theoretical probability. Assume that the coin is fair. Draw a bar-graph of the probability distribution (i.e. of the probability of getting getting 0, 1, 2, heads.)

8.4 Multiple Events and Combined Probabilities 3

A drawer contains 3 pairs of black socks, 2 pairs of white socks, 1 pair of green socks, and 2 pairs of blue socks. Two socks are pulled out at random from the drawer. (Assume each sock has the same probability of being selected).

(a) What is the probability that the pair pulled out consists of two black socks?

(b) What is the probability that the pair pulled out is either black or blue?

(c) What is the probability that a matching pair of any color is obtained?

(d) What is the probability that the pair pulled out of the drawer do not match?

8.5 Multiple Events and Combined Probabilities 4

A child's toy consists of a clear plastic box containing five coloured spheres (red, green, blue, yellow, and white) and five coloured sockets. After shaking the box, the spheres tend to randomly settle into the sockets one by one, with equal probability for a given sphere settling into any unoccupied socket. At the end of "one experiment" each sphere occupies exactly one socket, and all sockets are occupied.

(a) What is the probability that the red sphere will settle into the red socket?

(b) What is the probability that either the red sphere will settle into the red socket or the blue sphere will settle into the blue socket?

(c) What is the probability that the red sphere will settle into the red socket and also the blue sphere will settle into the blue socket?

(d) What is the probability that all the spheres settle into the matching sockets?

8.6 Expected Value and Probability

A coin was tossed 8 times by each person in a group of people. The number of people, $N(x)$, who got a total of x heads were as follows: $(x = 0, N = 0), (x = 1, N = 3), (x = 2, N = 10), (x = 3, N = 16), (x = 4, N = 25), (x = 5, N = 21), (x = 6, N = 6), (x = 7, N = 2), (x = 8, N = 2)$.

(a) Based on this experimental data, determine the (empirical) probability of obtaining x heads out of 8 coin tosses for $x = 0, 1, \ldots 8$.

(b) Find $\bar{x}$, the expected number of heads given the above data.

(c) Compare your results with the expected value of heads in a theoretical distribution in which the probability of H and T are equal.

8.7 Permutation and Combination 1

(a) Write out in full all possible outcomes of tossing 4 fair coins. How many such outcomes are there?

(b) Determine the probability of tossing 0, 1, 2, 3, 4 heads.

(c) Find the expected number of heads.

8.8 Permutation and combination 2

(a) Four athletes from four different countries compete in the final of 500 m speed skating. How many possible outcomes are there? (Assume that they all complete the race and that simultaneous arrival does not occur).

(b) How many ways are there to seat six people at a dining table with six fixed seats?

(c) How many ways are there to get a total of 8 by tossing two die simultaneously ? (consider that a 2 for dice #1 and a 6 for dice #2 is different from a 6 for dice #1 and a 2 for dice #2)

8.9 Permutation and Combination 3

Suppose you have 6 books and you want to put 3 on the book shelves. how many possible arrangements are there?

8.10 Permutation and Combination 4

Suppose you have 6 books again. In how many ways can you choose 3 to take on a trip.

8.11 Permutation and Combination 5

How many ways are there to get 3 times H and 2 times T by tossing a fair coin 5 times? What is the probability of getting 3 heads in 5 fair coin tosses?

8.12 Permutation and Combination 6

How many words can be made from the word "calculus"? (These words do not have to have any meaning. For examples, "alcuclus", "lcaluscu".)

8.13 The Binomial Theorem 1

(a) Find terms in Pascal's triangle down to the level that represents the coefficients $C(10, k)$.

(b) Use this to form the binomial expansion of the product

$$(p + q)^{10}$$

(c) What is the probability of getting exactly 7 heads in 10 tosses if the coin is fair?

(d) What is the probability of getting exactly 7 heads in 10 tosses if $p(H) = 0.49$?

(e) Draw the probability distribution for the probability of obtaining $m = 0, 1, 2, \cdots, 10$ heads when tossing a fair coin 10 times (i.e. $n = 10$). Recall that $2^{10} = 1024$.

8.14 Binomial Theorem 2

A biased coin has non-equal probabilities of tossing H or T. (P(H)=$\frac{3}{4}$ and P(T)=$\frac{1}{4}$) What is the probability of obtaining exactly 6 heads and 4 tails if you toss the coin 10 times.

8.15 Binomial Theorem 3

Suppose a shipment has 5 good items and 2 defective items. Select a sample of 3 and find the probability of exactly 2 good items and 1 defective item in the sample. Assume that the sampling is done with replacement (i.e. three times you randomly choose one item, check it, and return it).

8.16 Binomial Theorem 4

In order to construct a phylogenetic tree, it is necessary to trace evolutionary relationship between organisms. It is important to recognize what characteristics could be used to trace their evolutions. The most accepted characteristics used today is the structure of one of the RNA molecules making-up ribosomes. It has been found through nucleotide sequence comparisons of rRNA from many organisms, that some regions of rRNA are very similar in all organisms while others are quite variable. Evolutionary relationships can be determined using lines with lengths proportional to the number of differences between nucleotides. Two organisms, A and B, have been compared. The

probability for them to have the same nucleotides is $\frac{1}{5}$. If 5 nucleotides have been compared,what is the probability for having

(a) 3 different nucleotides?

(b) 4 different nucleotides?

(c) 5 different nucleotides?

(d) at most 2 are different?

8.17 Tossing a fair coin 8 times

Determine the probabilities of getting 1, 2, 3, or any number $k \leq 8$ of Heads when a fair coin is tossed 8 times. Give your results in the form of a table and draw the bar graph corresponding to the (theoretical) probability distribution.

8.18 Unfair coin

Suppose the coin is not fair, so that the probability of heads is $p = 0.45$. Determine the probability of getting exactly 3 heads in a total of 8 tosses.

8.19 The Cumulative distribution

Draw the cumulative distribution corresponding to the 8 coin toss experiment for a fair coin.

8.20

Given below is the distribution of the number of heads (H) obtained by a group of people in an experiment in which each person tossed the coin ten times. ($N(x)$=number of people who got x heads.) Use the spreadsheet to plot the (empirical) probability distribution of obtaining x heads in 10 tosses based on this data and the cumulative distribution (of obtaining up to k heads) on the same graph. Use the spreadsheet to calculate the expected number of heads based on the same data. Submit a graph of the probability distribution and cumulative distribution on which the calculated expected value is given.

num heads	0	1	2	3	4	5	6	7	8	9	10
num people	1	1	4	15	17	25	21	13	3	2	1

8.21 Moments of a discrete distribution

Recall that the j'th moment, M_j of a distribution is

$$M_j = \sum_{i=0}^{N} (x_i)^j p(x_i).$$

Show that for a discrete probability distribution, $M_0 = 1$, and M_1 corresponds to the mean, $\bar{x}$. Then show that the Variance can be expressed as

$$V = M_2 - (\bar{x})^2$$

8.22

A hat contains 21 identical tags numbered 1, 2, 3, .. 21. (The numbers are printed on the tags, and each number in the list occurs only once.) You select one tag at random from the hat, without looking. (Assume that each tag has an equal probability of being picked.) Let x = the number on the tag you have picked

(a) Determine the expected value of x.

(b) Find the variance of x.

(c) What is the probability of selecting a tag with an odd number?