Chapter 10

Infinite series, improper integrals, and Taylor series

10.1 Introduction

This chapter has several important and challenging goals. The first of these is to understand how concepts that were discussed for finite series and integrals can be meaningfully extended to infinite series and improper integrals - i.e. integrals of functions over an infinite domain. In this part of the discussion, we will find that the notion of convergence and divergence will be important.

A second theme will be that of approximation of functions in terms of power series, also called Taylor series. Such series can be described informally as infinite polynomials (i.e. polynomials containing infinitely many terms). Understanding when these objects are meaningful is also related to the issue of convergence, so we use the background assembled in the first part of the chapter to address such concepts arising in the second part.

Figure 10.1. The function \( y = f(x) \) (solid heavy curve) is shown together with its linear approximation (LA, dashed line) at the point \( x_0 \), and a better “higher order” approximation (HOA, thin solid curve). Notice that this better approximation stays closer to the graph of the function near \( x_0 \). In this chapter, we discuss how such better approximations can be obtained.

The theme of approximation has appeared often in our calculus course. In a previous
semester, we discussed a **linear approximation** to a function. The idea was to approximate the value of the function close to a point on its graph using a straight line (the tangent line). We noted in doing so that the approximation was good only close to the point of tangency. Further away, the graph of the functions curves away from that straight line. This leads naturally to the question: can we do better in making this approximation if we include other terms to describe this “curving away”? Here we extend such linear approximation methods. Our goal is to increase the accuracy of the linear approximation by including higher order terms (quadratic, cubic, etc), i.e. to find a polynomial that approximates the given function. This idea forms an important goal in this chapter.

We first review the idea of series introduced in Chapter 1.

### 10.2 Convergence and divergence of series

Recall the geometric series discussed in Section 1.6.

The sum of a **finite geometric series**, 

\[ S_n = 1 + r + r^2 + \ldots + r^n = \sum_{k=0}^{n} r^k, \quad \text{is} \quad S_n = \frac{1 - r^{n+1}}{1 - r}. \tag{10.1} \]

We also review definitions discussed in Section 1.7

**Definition: Convergence of infinite series**

An infinite series that has a finite sum is said to be **convergent**. Otherwise it is **divergent**.

**Definition: Partial sums and convergence**

Suppose that \( S \) is an (infinite) series whose terms are \( a_k \). Then the **partial sums**, \( S_n \), of this series are

\[ S_n = \sum_{k=0}^{n} a_k. \]

We say that the sum of the infinite series is \( S \), and write

\[ S = \sum_{k=0}^{\infty} a_k, \quad \text{provided that} \quad S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=0}^{n} a_k. \]

That is, we consider the infinite series as the limit of partial sums \( S_n \) as the number of terms \( n \) is increased. If this limit exists, we say that the infinite series **converges**\(^{58}\) to \( S \). This leads to the following conclusion:

\(^{58}\) If the limit does not exist, we say that the series diverges.
10.2. Convergence and divergence of series

The sum of an infinite geometric series,

\[ S = 1 + r + r^2 + \ldots + r^k + \ldots = \sum_{k=0}^{\infty} r^k = \frac{1}{1 - r}, \quad \text{provided } |r| < 1. \quad (10.2) \]

If this inequality is not satisfied, then we say that this sum does not exist (meaning that it is not finite).

It is important to remember that an infinite series, i.e. a sum with infinitely many terms added up, can exhibit either one of these two very different behaviours. It may converge in some cases, as the first example shows, or diverge (fail to converge) in other cases. We will see examples of each of these trends again. It is essential to be able to distinguish the two. Divergent series (or series that diverge under certain conditions) must be handled with particular care, for otherwise, we may find contradictions or "seemingly reasonable" calculations that have meaningless results.

We can think of convergence or divergence of series using a geometric analogy. If we start on the number line at the origin and take a sequence of steps \( \{a_0, a_1, a_2, \ldots, a_k, \ldots\} \), we can think of \( S = \sum_{k=0}^{\infty} a_k \) as the total distance we have travelled. \( S \) converges if that distance remains finite and if we approach some fixed number.

\[ "convergence" \]

\[ "divergence" \]

**Figure 10.2.** An informal schematic illustrating the concept of convergence and divergence of infinite series. Here we show only a few terms of the infinite series: from left to right, each step is a term in the series. In the top example, the sum of the steps gets closer and closer to some (finite) value. In the bottom example, the steps lead to an ever increasing total sum.

In order for the sum of ‘infinitely many things’ to add up to a finite number, the terms have to get smaller. But just getting smaller is not, in itself, enough to guarantee convergence. (We will show this later on by considering the harmonic series.) There are rigorous mathematical tests which help determine whether a series converges or not. We discuss some of these tests in Appendix 11.9.
10.3 Improper integrals

We will see that there is a close connection between certain infinite series and improper integrals, i.e. integrals over an infinite domain. We have already encountered an example of an improper integral in Section 3.8.5 and in the context of radioactive decay in Section 8.4.

Recall the following definition:

**Definition**

An improper integral is an integral performed over an infinite domain, e.g.

\[ \int_a^\infty f(x) \, dx. \]

The value of such an integral is understood to be a limit, as given in the following definition:

\[ \int_a^\infty f(x) \, dx = \lim_{b \to \infty} \int_a^b f(x) \, dx. \]

i.e. we evaluate an improper integral by first computing a definite integral over a finite domain \( a \leq x \leq b \), and then taking a limit as the endpoint \( b \) moves off to larger and larger values. The definite integral can be interpreted as an area under the graph of the function.

The essential question being addressed here is whether that area remains bounded when we include the “infinite tail” of the function (i.e. as the endpoint \( b \) moves to larger values.) For some functions (whose values get small enough fast enough) the answer is “yes”.

**Definition**

When the limit shown above exists, we say that the improper integral converges. Otherwise we say that the improper integral diverges.

With this in mind, we compute a number of classic integrals:

10.3.1 Example: A decaying exponential: convergent improper integral

Here we recall that the improper integral of a decaying exponential converges. (We have seen this earlier, in Section 3.8.5, and again in applications in Sections 4.5 and 8.4.1. Here we recap this important result in the context of our discussion of improper integrals.) Suppose that \( r > 0 \) and let

\[ I = \int_0^\infty e^{-rt} \, dt \equiv \lim_{b \to \infty} \int_0^b e^{-rt} \, dt. \]

Then note that \( b > 0 \) so that

\[ I = \lim_{b \to \infty} \left[ -\frac{1}{r} e^{-rt} \right]_0^b = -\frac{1}{r} \lim_{b \to \infty} (e^{-rb} - e^0) = -\frac{1}{r} (\lim_{b \to \infty} e^{-rb} - 1) = \frac{1}{r}. \]
We have used the fact that \( \lim_{b \to \infty} e^{-rb} = 0 \) since (for \( r, b > 0 \)) the exponential function is decreasing with increasing \( b \). Thus the limit exists (is finite) and the integral converges. In fact it converges to the value \( I = 1/r \).

\[ I = \int_{1}^{\infty} \frac{1}{x} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x} \, dx = \lim_{b \to \infty} \ln(x) \bigg|_{1}^{b} = \lim_{b \to \infty} (\ln(b) - \ln(1)) \]

\[ I = \lim_{b \to \infty} \ln(b) = \infty \]

The fact that we get an infinite value for this integral follows from the observation that \( \ln(b) \) increases without bound as \( b \) increases, that is the limit does not exist (is not finite). Thus the area under the curve \( f(x) = 1/x \) over the interval \( 1 \leq x \leq \infty \) is infinite. We say that the improper integral of \( 1/x \) diverges (or does not converge). We will use this result again in Section 10.4.1.

59We do not chose the interval \((0, \infty)\) because this function is undefined at \( x = 0 \). We want here to emphasize the behaviour at infinity, not the blow up that occurs close to \( x = 0 \).
10.3.3 Example: The improper integral of $1/x^2$ converges

Now consider the related function

$$y = f(x) = \frac{1}{x^2}, \quad \text{and the corresponding integral} \quad I = \int_1^\infty \frac{1}{x^2} \, dx$$

Then

$$I = \lim_{b \to \infty} \int_1^b x^{-2} \, dx = \left. \lim_{b \to \infty} (-x^{-1}) \right|_1^b = -\lim_{b \to \infty} \left( \frac{1}{b} - 1 \right) = 1.$$

Thus, the limit exists, and, in fact, $I = 1$, so, in contrast to the Example 10.3.2, this integral converges.

We observe that the behaviours of the improper integrals of the functions $1/x$ and $1/x^2$ are very different. The former diverges, while the latter converges. The only difference between these functions is the power of $x$. As shown in Figure 10.3, that power affects how rapidly the graph “falls off” to zero as $x$ increases. The function $1/x^2$ decreases much faster than $1/x$. (Consequently $1/x^2$ has a sufficiently “slim” infinite “tail”, that the area under its graph does not become infinite - not an easy concept to digest!) This observations leads us to wonder what power $p$ is needed to make the improper integral of a function $1/x^p$ converge. We answer this question below.

10.3.4 When does the integral of $1/x^p$ converge?

Here we consider an arbitrary power, $p$, that can be any real number. We ask when the corresponding improper integral converges or diverges. Let

$$I = \int_1^\infty \frac{1}{x^p} \, dx.$$

For $p = 1$ we have already established that this integral diverges (Section 10.3.2), and for $p = 2$ we have seen that it is convergent (Section 10.3.3). By a similar calculation, we find that

$$I = \lim_{b \to \infty} \frac{x^{1-p}}{(1-p)} \bigg|_1^b = \lim_{b \to \infty} \left( \frac{1}{1-p} \right) (b^{1-p} - 1).$$

Thus this integral converges provided that the term $b^{1-p}$ does not “blow up” as $b$ increases. For this to be true, we require that the exponent $(1-p)$ should be negative, i.e. $1 - p < 0$ or $p > 1$. In this case, we have

$$I = \frac{1}{p-1}.$$

To summarize our result,

$$\int_1^\infty \frac{1}{x^p} \, dx \quad \text{converges if } p > 1, \quad \text{diverges if } p \leq 1.$$
10.3. Improper integrals

Examples: \( \int 1/x^p \) that do or do not converge

1. The integral

\[
\int_1^\infty \frac{1}{\sqrt{x}} \, dx,
\]

diverges. We see this from the following argument: \( \sqrt{x} = x^{1/2} \), so \( p = \frac{1}{2} < 1 \). Thus, by the general result, this integral diverges.

2. The integral

\[
\int_1^\infty x^{-1.01} \, dx,
\]

converges. Here \( p = 1.01 > 1 \), so the result implies convergence of the integral.

10.3.5 Integral comparison tests

The integrals discussed above can be used to make comparisons that help us to identify when other improper integrals converge or diverge\(^{60}\). The following important result establishes how these comparisons work:

\[a\]

Suppose we are given two functions, \( f(x) \) and \( g(x) \), both continuous on some infinite interval \([a, \infty)\). Suppose, moreover, that, at all points on this interval, the first function is smaller than the second, i.e.

\[0 \leq f(x) \leq g(x)\]

Then the following conclusions can be made:

1. \( \int_a^\infty f(x) \, dx \leq \int_a^\infty g(x) \, dx \). (This means that the area under \( f(x) \) is smaller than the area under \( g(x) \).)

2. If \( \int_a^\infty g(x) \, dx \) converges, then \( \int_a^\infty f(x) \, dx \) converges. (If the larger area is finite, so is the smaller one)

3. If \( \int_a^\infty f(x) \, dx \) diverges, then \( \int_a^\infty g(x) \, dx \) diverges. (If the smaller area is infinite, so is the larger one.)

\(^{60}\)These statements have to be carefully noted. What is assumed and what is concluded works “one way”. That is the order “if..then” is important. Reversing that order leads to a common error.

\[^{60}\]The reader should notice the similarity of these ideas to the comparisons made for infinite series in the Appendix 11.9.2. This similarity stems from the fact that there is a close connection between series and integrals, a recurring theme in this course.
Example: comparison of improper integrals

We can determine that the integral
\[ \int_1^\infty \frac{x}{1 + x^3} \, dx \]
converges by noting that for all \( x > 0 \)

\[ 0 \leq \frac{x}{1 + x^3} \leq \frac{x}{x^3} = \frac{1}{x^2}. \]

Thus
\[ \int_1^\infty \frac{x}{1 + x^3} \, dx \leq \int_1^\infty \frac{1}{x^2} \, dx. \]

Since the larger integral on the right is known to converge, so does the smaller integral on the left.

10.4 Comparing integrals and series

The convergence of infinite series was discussed earlier, in Section 1.7 and here again in Section 10.2. Many tests for convergence are provided in the Appendix 11.9, and will not be discussed in detail due to time and space constraints. However, an interesting connection exists between convergence of series and integrals. This is the topic we examine here.

Convergence of series and convergence of integrals is related. We can use the convergence/divergence of an integral/series to determine the behaviour of the other. Here we give an example of this type by establishing a connection between the harmonic series and a divergent improper integral.

10.4.1 The harmonic series

The harmonic series is a sum of terms of the form \( 1/k \) where \( k = 1, 2, \ldots \). At first appearance, this series might seem to have the desired qualities of a convergent series, simply because the successive terms being added are getting smaller and smaller, but this appearance is deceptive and actually wrong. We establish that the harmonic series diverges by comparing it to the improper integral of the related function.

\[
\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{k} + \ldots \quad \text{diverges}
\]

We have already noticed a similar surprise in connection with the improper integral of \( 1/x \). These two “surprises” are closely related, as we show here using a comparison of the series and the integral.

This function is “related” since for integer values of \( x \), the function takes on values that are the same as successive terms in the series, i.e. if \( x = k \) is an integer, then \( f(x) = f(k) = 1/k \).
Figure 10.4. The harmonic series is a sum that corresponds to the area under the staircase shown above. Note that we have purposely shown the stairs arranged so that they are higher than the function. This is essential in drawing the conclusion that the sum of the series is infinite: It is larger than an area under $1/x$ that we already know to be infinite, by Section 10.3.2.

In Figure 10.4 we show on one graph a comparison of the area under this curve, and a staircase area representing the first few terms in the harmonic series. For the area of the staircase, we note that the width of each step is 1, and the heights form the sequence

$$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$$

Thus the area of (infinitely many) of these steps can be expressed as the (infinite) harmonic series,

$$A = 1 \cdot 1 + 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{4} + \ldots = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = \sum_{k=1}^{\infty} \frac{1}{k}.$$  

On the other hand, the area under the graph of the function $y = f(x) = 1/x$ for $0 \leq x \leq \infty$ is given by the improper integral

$$\int_{1}^{\infty} \frac{1}{x} \, dx.$$
We have seen previously in Section 10.3.2 that this integral diverges!
From Figure 10.4 we see that the areas under the function, \( A_f \) and under the staircase, \( A_s \), satisfy
\[
0 < A_f < A_s.
\]
Thus, since the smaller of the two (the improper integral) is infinite, so is the larger (the sum of the harmonic series). We have established, using this comparison, that the sum of the harmonic series cannot be finite, so that this series diverges.

**Other comparisons: The “p” series**

More generally, we can compare series of the form
\[
\sum_{k=1}^{\infty} \frac{1}{k^p}
\]
to the integral
\[
\int_1^{\infty} \frac{1}{x^p} \, dx
\]
in precisely the same way. This leads to the conclusion that

| The “p” series, \( \sum_{k=1}^{\infty} \frac{1}{k^p} \) | converges if \( p > 1 \), diverges if \( p \leq 1 \). |

For example, the series
\[
\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots
\]
converges, since \( p = 2 > 1 \). Notice, however, that the comparison does not give us a value to which the sum converges. It merely indicates that the series does converge.

**10.5 From geometric series to Taylor polynomials**

In studying calculus, we explored a variety of functions. Among the most basic are polynomials, i.e. functions such as
\[
p(x) = x^5 + 2x^2 + 3x + 2.
\]
Our introduction to differential calculus started with such functions for a reason: these functions are convenient and simple to handle. We found long ago that it is easy to compute derivatives of polynomials. The same can be said for integrals. One of our first examples, in Section 3.6.1 was the integral of a polynomial. We needed only use a power rule to integrate each term. An additional convenience of polynomials is that “evaluating’ the function, (i.e. plugging in an \( x \) value and determining the corresponding \( y \) value) can be done by simple multiplications and additions, i.e. by basic operations easily handled by an ordinary calculator. This is not the case for, say, trigonometric functions, exponential
functions, or for that matter, most other functions we considered\textsuperscript{63}. For this reason, being able to \emph{approximate} a function by a polynomial is an attractive proposition. This forms our main concern in the sections that follow.

We can arrive at connections between several functions and their polynomial approximations by exploiting our familiarity with the \textit{geometric series}. We use both the results for convergence of the geometric series (from Section 10.2) and the formula for the sum of that series to derive a number of interesting, (somewhat haphazard) results\textsuperscript{64}.

Recall from Sections 1.7.1 and 10.2 that the \textit{sum of an infinite geometric series is}

\begin{equation}
S = 1 + r + r^2 + \ldots + r^k + \ldots = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \quad \text{provided } |r| < 1.
\end{equation}

To connect this result to a statement about a function, we need a “variable”. Let us consider the behaviour of this series when we vary the quantity $r$. To emphasize that now $r$ is our variable, it will be helpful to change notation by substituting $r = x$ into the above equation, while remembering that the formula in Eqn (10.3) hold only provided $|r| = |x| < 1$.

\subsection*{10.5.1 Example 1: A simple expansion}

Substitute the variable $x = r$ into the series (10.3). Then formally, rewriting the above with this substitution, leads to the conclusion that

\begin{equation}
\frac{1}{1-x} = 1 + x + x^2 + \ldots
\end{equation}

We can think of this result as follows: Let

\begin{equation}
f(x) = \frac{1}{1-x}
\end{equation}

Then for every $x$ in $-1 < x < 1$, it is true that $f(x)$ can be approximated by terms in the polynomial

\begin{equation}
p(x) = 1 + x + x^2 + \ldots
\end{equation}

In other words, by (10.3), for $|x| < 1$ the two expressions “are the same”, in the sense that the polynomial converges to the value of the function. We refer to this $p(x)$ as an (infinite) Taylor polynomial\textsuperscript{65} or simply a \textit{Taylor series} for the function $f(x)$. The usefulness of this kind of result can be illustrated by a simple example.

\textbf{Example 10.1 (Using the Taylor Series (10.6) to approximate the function (10.5))} Compute the value of the function $f(x)$ given by Eqn. (10.5) for $x = 0.1$ without using a calculator.

\textsuperscript{63}For example, to find the decimal value of $\sin(2.5)$ we would need a scientific calculator. These days the distinction is blurred, since powerful hand-held calculators are ubiquitous. Before such devices were available, the ease of evaluating polynomials made them even more important.

\textsuperscript{64}We say “haphazard” here because we are not yet at the point of a systematic procedure for computing a Taylor Series. That will be done in Section 10.6. Here we “take what we can get” using simple manipulations of a geometric series.

\textsuperscript{65}A Taylor polynomial contains finitely many terms, $n$, whereas a Taylor series has $n \to \infty$. 

Solution: Plugging in the value \( x = 0.1 \) into the function directly leads to \( 1/(1 - 0.1) = 1/0.9 \), whose evaluation with no calculator requires long division. Using the polynomial representation, we have a simpler method:

\[
p(0.1) = 1 + 0.1 + 0.1^2 + \ldots = 1 + 0.1 + 0.01 + \ldots = 1.11\ldots
\]

We provide a few other examples based on substitutions of various sorts using the geometric series as a starting point.

10.5.2 Example 2: Another substitution

We make the substitution \( r = -t \), then \( |r| < 1 \) means that \( |-t| = |t| < 1 \), so that the formula (10.3) for the sum of a geometric series implies that:

\[
\frac{1}{1 - (-t)} = 1 + (-t) + (-t)^2 + (-t)^3 + \ldots
\]

\[
\frac{1}{1 + t} = 1 - t + t^2 - t^3 + t^4 + \ldots \quad \text{provided } |t| < 1
\]

This means we have produced a series expansion for the function \( 1/(1 + t) \). We can go farther with this example by a new manipulation, whereby we integrate both sides to arrive at a new function and its expansion, shown next.

10.5.3 Example 3: An expansion for the logarithm

We will use the results of Example 10.5.2, but we follow our substitution by integration. On the left, we integrate the function \( f(t) = 1/(1 + t) \) (to arrive at a logarithm type integral) and on the right we integrate the power terms of the expansion. We are permitted to integrate the power series term by term provided that the series converges. This is an important restriction that we emphasize: Manipulation of infinite series by integration, differentiation, addition, multiplication, or any other “term by term” computation makes sense only so long as the original series converges.

Provided \( |t| < 1 \), we have that

\[
\int_0^x \frac{1}{1 + t} \, dt = \int_0^x (1 - t + t^2 - t^3 + t^4 - \ldots) \, dt
\]

\[
\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots
\]

This procedure has allowed us to find a series representation for a new function, \( \ln(1 + x) \).

\[
\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}. \quad (10.7)
\]

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\[\text{This example is slightly “trivial”, in the sense that evaluating the function itself is not very difficult. However, in other cases, we will find that the polynomial expansion is the only way to find the desired value.}\]
The formula appended on the right is just a compact notation that represents the pattern of the terms. Recall that in Chapter 1, we have gotten thoroughly familiar with such summation notation.\(^{67}\)

**Example 10.2 (Evaluating the logarithm for \(x = 0.25\))** An expansion for the logarithm is definitely useful, in the sense that (without a scientific calculator or log tables) it is not possible to easily calculate the value of this function at a given point. For example, for \(x = 0.25\), we cannot find \(\ln(1 + 0.25) = \ln(1.25)\) using simple operations, whereas the value of the first few terms of the series are computable by simple multiplication, division, and addition \((0.25 - \frac{0.25^2}{2} + \frac{0.25^3}{3} \approx 0.2239)\). (A scientific calculator gives \(\ln(1.25) \approx 0.2231\), so the approximation produced by the series is relatively good.)

When is the series for \(\ln(1 + x)\) in (10.7) expected to converge? Retracing our steps from the beginning of Example 10.5.2 we note that the value of \(t\) is not permitted to leave the interval \(|t| < 1\) so we need also \(|x| < 1\) in the integration step.\(^{68}\) We certainly cannot expect the series for \(\ln(1 + x)\) to converge when \(|x| > 1\). Indeed, for \(x = -1\), we have \(\ln(1 + x) = \ln(0)\) which is undefined. Also note that for \(x = -1\) the right hand side of (10.7) becomes

\[-\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right).\]

This is the recognizable harmonic series (multiplied by -1). But we already know from Section 10.4.1 that the harmonic series diverges. Thus, we must avoid \(x = -1\), since the expansion will not converge there, and neither is the function defined. *This example illustrates that outside the interval of convergence, the series and the function become “meaningless”.*

**Example 10.3 (An expansion for \(\ln(2)\))** Strictly speaking, our analysis does not predict what happens if we substitute \(x = 1\) into the expansion of the function found in Section 10.5.3, because this value of \(x\) is outside of the permitted range \(-1 < x < 1\) in which the Taylor series can be guaranteed to converge. It takes some deeper mathematics (Abel’s theorem) to prove that the result of this substitution actually makes sense, and converges, i.e. that

\[
\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots
\]

We state without proof here that the **alternating harmonic series converges to \(\ln(2)\).**

**10.5.4 Example 4: An expansion for \(\arctan\)**

Suppose we make the substitution \(r = -t^2\) into the geometric series formula, and recall that we need \(|r| < 1\) for convergence. Then

\[
\frac{1}{1 - (-t^2)} = 1 + (-t^2) + (-t^2)^2 + (-t^2)^3 + \ldots
\]

\(^{67}\)The summation notation is not crucial and should certainly not be memorized. We are usually interested only in the first few terms of such a series in any approximation of practical value.

\(^{68}\)Strictly speaking, we should have ensured that we are inside this interval of convergence before we computed the last example.
\[
\frac{1}{1 + t^2} = 1 - t^2 + t^4 - t^6 + t^8 + \ldots = \sum_{k=0}^{\infty} (-1)^n t^{2n}
\]

This series will converge provided \(|t| < 1\). Now integrate both sides, and recall that the antiderivative of the function \(1/(1 + t^2)\) is \(\arctan(t)\). Then

\[
\int_0^x \frac{1}{1 + t^2} \, dt = \int_0^x (1 - t^2 + t^4 - t^6 + t^8 + \ldots) \, dt
\]

\[
\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{2k-1}, \quad (10.8)
\]

**Example 10.4 (An expansion for \(\pi\))** For a particular application of this expansion, consider plugging in \(x = 1\) into Equation (10.8). Then

\[
\arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots
\]

But \(\arctan(1) = \pi/4\). Thus we have found a way of computing the irrational number \(\pi\), namely

\[
\pi = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) = 4 \left(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{2k-1}\right).
\]

While it is true that this series converges, the convergence is slow. (This can be seen by adding up the first 100 or 1000 terms of this series with a spreadsheet.) This means that it is not practical to use such a series as an approximation for \(\pi\). (There are other series that converge to \(\pi\) very rapidly that are used in any practical application.)

### 10.6 Taylor Series: a systematic approach

In Section 10.5, we found a variety of Taylor series expansions directly from the formula for a geometric series. Here we ask how such Taylor series can be constructed more systematically, if we are given a function that we want to approximate \(^{69}\).

Suppose we have a function which we want to represent by a power series,

\[
f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_{k=0}^{\infty} a_k x^k.
\]

Here we will use the function to directly determine the coefficients \(a_k\). To determine \(a_0\), let \(x = 0\) and note that

\[
f(0) = a_0 + a_1 0 + a_2 0^2 + a_3 0^3 + \ldots = a_0.
\]

We conclude that

\[
a_0 = f(0).
\]

\(^{69}\)The development of this section was motivated by online notes by David Austin.
By differentiating both sides we find the following:

\[ f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + ka_k x^{k-1} + \ldots \]

\[ f''(x) = 2a_2 + 2 \cdot 3a_3 x + \ldots + (k-1)ka_k x^{k-2} + \ldots \]

\[ f'''(x) = 2 \cdot 3a_3 + \ldots + (k-2)(k-1)ka_k x^{k-3} + \ldots \]

\[ f^{(k)}(x) = 1 \cdot 2 \cdot 3 \ldots ka_k + \ldots \]

Here we have used the notation \( f^{(k)}(x) \) to denote the \( k \)’th derivative of the function. Now evaluate each of the above derivatives at \( x = 0 \). Then

\[ f'(0) = a_1, \quad \Rightarrow a_1 = f'(0) \]

\[ f''(0) = 2a_2, \quad \Rightarrow a_2 = \frac{f''(0)}{2} \]

\[ f'''(0) = 2 \cdot 3a_3, \quad \Rightarrow a_3 = \frac{f'''(0)}{2 \cdot 3} \]

\[ f^{(k)}(0) = k!a_k, \quad \Rightarrow a_k = \frac{f^{(k)}(0)}{k!} \]

This gives us a recipe for calculating all coefficients \( a_k \). This means that if we can compute all the derivatives of the function \( f(x) \), then we know the coefficients of the Taylor series as well. Because we have evaluated all the coefficients by the substitution \( x = 0 \), we say that the resulting power series is the Taylor series of the function about \( x = 0 \).

### 10.6.1 Taylor series for the exponential function, \( e^x \)

Consider the function \( f(x) = e^x \). All the derivatives of this function are equal to \( e^x \). In particular,

\[ f^{(k)}(x) = e^x \quad \Rightarrow \quad f^{(k)}(0) = 1. \]

So that the coefficients of the Taylor series are

\[ a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{k!} \]

Therefore the Taylor series for \( e^x \) about \( x = 0 \) is

\[ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_k x^k + \ldots = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots + \frac{x^k}{k!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \]

This is a very interesting series. We state here without proof that this series converges for all values of \( x \). Further, the function defined by the series is in fact equal to \( e^x \) that is,

\[
e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}\]
The implication is that the function \( e^x \) is completely determined (for all \( x \) values) by its behaviour (i.e. derivatives of all orders) at \( x = 0 \). In other words, the value of the function at \( x = 1,000,000 \) is determined by the behaviour of the function around \( x = 0 \). This means that \( e^x \) is a very special function with superior "predictable features". If a function \( f(x) \) agrees with its Taylor polynomial on a region \((-a, a)\), as was the case here, we say that \( f \) is analytic on this region. It is known that \( e^x \) is analytic for all \( x \).

We can use the results of this example to establish the fact that the exponential function grows "faster" than any power function \( x^n \). That is the same as saying that the ratio of \( e^x \) to \( x^n \) (for any power \( n \)) increases with \( x \). We leave this as an exercise for the reader.

We can also easily obtain a Taylor series expansion for functions related to \( e^x \), without assembling the derivatives. We start with the result that

\[
e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \ldots = \sum_{k=0}^{\infty} \frac{u^k}{k!}
\]

Then, for example, the substitution \( u = x^2 \) leads to

\[
e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2} + \frac{(x^2)^3}{6} + \ldots = \sum_{k=0}^{\infty} \frac{(x^2)^k}{k!}
\]

### 10.6.2 Taylor series of trigonometric functions

In this example we determine the Taylor series for the sine function. The function and its derivatives are

\[
f(x) = \sin x, f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x, f^{(4)}(x) = \sin x, \ldots
\]

After this, the cycle repeats. This means that

\[
f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1, \ldots
\]

and so on in a cyclic fashion. In other words,

\[
a_0 = 0, a_1 = 1, a_2 = 0, a_3 = -\frac{1}{3!}, a_4 = 0, a_5 = \frac{1}{5!}, \ldots
\]

Thus,

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.
\]

We state here without proof that the function \( \sin(x) \) is analytic, so that the expansion converges to the function for all \( x \).

It is instructive to demonstrate how successive terms in a Taylor series expansion lead to approximations that improve. Doing this kind of thing will be the subject of the last computer laboratory exercise in this course.
Figure 10.5. An approximation of the function $y = \sin(x)$ by successive Taylor polynomials, $T_1, T_2, T_3, T_4$. The higher Taylor polynomials do a better job of approximating the function on a larger interval about $x = 0$.

Here we demonstrate this idea with the expansion for the function $\sin(x)$ that we just obtained. To see this, consider the sequence of polynomials

- $T_1(x) = x$,
- $T_2(x) = x - \frac{x^3}{3!}$,
- $T_3(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$,
- $T_4(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$.

Then these polynomials provide a better and better approximation to the function $\sin(x)$ close to $x = 0$. The first of these is just a linear (or tangent line) approximation that we had studied long ago. The second improves this with a quadratic approximation, etc. Figure 10.5 illustrates how the first few Taylor polynomials approximate the function $\sin(x)$ near $x = 0$. Observe that as we keep more terms, $n$, in the polynomial $T_n(x)$, the approximating curve “hugs” the graph of $\sin(x)$ over a longer and longer range. The student will be asked to use the spreadsheet, together with some calculations as done in this section, to produce a composite graph similar to Fig. 10.5 for some other function.
Example 10.5 (The error in successive approximations) For a given value of \( x \) close to the base point (at \( x = 0 \)), the error in the approximation between the polynomials and the function is the vertical distance between the graphs of the polynomial and the function \( \sin(x) \) (shown in black). For example, at \( x = 2 \) radians \( \sin(2) = 0.9093 \) (as found on a scientific calculator). The approximations are: \( T_1(2) = 2 \), which is very inaccurate, \( T_2(2) = 2 - \frac{2^3}{3!} \approx 0.667 \) which is too small, \( T_3(2) \approx 0.9333 \) that is much closer and \( T_4(2) \approx .9079 \) that is closer still. In general, we can approximate the size of the error using the next term that would occur in the polynomial if we kept a higher order expansion. The details of estimating such errors is omitted from our discussion.

We also note that all polynomials that approximate \( \sin(x) \) contain only odd powers of \( x \). This stems from the fact that \( \sin(x) \) is an odd function, i.e. its graph is symmetric to rotation about the origin, a concept we discussed in an earlier term.

The Taylor series for \( \cos(x) \) could be found by a similar sequence of steps. But in this case, this is unnecessary: We already know the expansion for \( \sin(x) \), so we can find the Taylor series for \( \cos(x) \) by simple differentiation term by term. (Since \( \sin(x) \) is analytic, this is permitted for all \( x \).) We leave as an exercise for the reader to show that

\[
\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.
\]

Since \( \cos(x) \) has symmetry properties of an even function, we find that its Taylor series is composed of even powers of \( x \) only.

10.7 Application of Taylor series

In this section we illustrate some of the applications of Taylor series to problems that may be difficult to solve using other conventional methods. Some functions do not have an antiderivative that can be expressed in terms of other simple functions. Integrating these functions can be a problem, as we cannot use the Fundamental Theorem of Calculus specifies. In some cases, we can approximate the value of the definite integral using a Taylor series. We show this in Section 10.7.1.

Another application of Taylor series is to compute an approximate solution to a differential equation. We provide one example of that sort in Section 10.7.2 and another in Appendix 11.11.

10.7.1 Example 1: using a Taylor series to evaluate an integral

Evaluate the definite integral

\[
\int_0^1 \sin(x^2) \, dx.
\]

A simple substitution (e.g. \( u = x^2 \)) will not work here, and we cannot find an antiderivative. Here is how we might approach the problem using Taylor series: We know that the
10.7. Application of Taylor series

Series expansion for \( \sin(t) \) is

\[
\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots
\]

Substituting \( t = x^2 \), we have

\[
\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots
\]

In spite of the fact that we cannot antidifferentiate the function, we can antidifferentiate the Taylor series, just as we would a polynomial:

\[
\int_0^1 \sin(x^2) \, dx = \int_0^1 \left( x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots \right) \, dx
\]

\[
= \left. \left( \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \ldots \right) \right|_0^1
\]

\[
= \frac{1}{3} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!} - \frac{1}{15 \cdot 7!} + \ldots
\]

This is an alternating series so we know that it converges. If we add up the first four terms, the pattern becomes clear: the series converges to 0.31026.

10.7.2 Example 2: Series solution of a differential equation

We are already familiar with the differential equation and initial condition that describes unlimited exponential growth.

\[
\frac{dy}{dx} = y, \quad y(0) = 1.
\]

Indeed, from previous work, we know that the solution of this differential equation and initial condition is \( y(x) = e^x \), but we will pretend that we do not know this fact in illustrating the usefulness of Taylor series. In some cases, where separation of variables does not work, this option would have great practical value.

Let us express the “unknown” solution to the differential equation as

\[
y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots
\]

Our task is to determine values for the coefficients \( a_i \).

Since this function satisfies the condition \( y(0) = 1 \), we must have \( y(0) = a_0 = 1 \). Differentiating this power series leads to

\[
\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots
\]
But according to the differential equation, $\frac{dy}{dx} = y$. Thus, it must be true that the two Taylor series match, i.e.

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots$$

This equality holds for all values of $x$. This can only happen if the coefficients of like terms are the same, i.e. if the constant terms on either side of the equation are equal, if the terms of the form $C x^2$ on either side are equal, and so on for all powers of $x$. Equating coefficients, we obtain:

- $a_0 = a_1 = 1$, $\Rightarrow a_1 = 1$,
- $a_1 = 2a_2$, $\Rightarrow a_2 = \frac{a_1}{2} = \frac{1}{2}$,
- $a_2 = 3a_3$, $\Rightarrow a_3 = \frac{a_2}{3} = \frac{1}{6}$,
- $a_3 = 4a_4$, $\Rightarrow a_4 = \frac{a_3}{4} = \frac{1}{24}$,
- $a_{n-1} = na_n$, $\Rightarrow a_n = \frac{a_{n-1}}{n} = \frac{1}{1 \cdot 2 \cdot 3 \ldots n} = \frac{1}{n!}$,

This means that

$$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots = e^x,$$

which, as we have seen, is the expansion for the exponential function. This agrees with the solution we have been expecting. In the example here shown, we would hardly need to use series to arrive at the right conclusion, but in the next example, we would not find it as easy to discover the solution by other techniques discussed previously.

We provide an example of a more complicated differential equation and its series solution in Appendix 11.11.

### 10.8 Summary

The main points of this chapter can be summarized as follows:

1. We reviewed the definition of an improper integral

$$\int_a^\infty f(x) \, dx = \lim_{b \to \infty} \int_a^b f(x) \, dx.$$  

2. We computed some examples of improper integrals and discussed their convergence or divergence. We recalled (from earlier chapters) that

$$I = \int_0^\infty e^{-rt} \, dt \quad \text{converges},$$

whereas

$$I = \int_1^\infty \frac{1}{x} \, dx \quad \text{diverges}.$$
3. More generally, we showed that
\[ \int_1^\infty \frac{1}{x^p} \, dx \quad \text{converges if } p > 1, \text{ diverges if } p \leq 1. \]

4. Using a comparison between integrals and series we showed that the harmonic series,
\[ \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{k} + \ldots \quad \text{diverges.} \]

5. More generally, our results led to the conclusion that the “p” series,
\[ \sum_{k=1}^{\infty} \frac{1}{k^p} \quad \text{converges if } p > 1, \text{ diverges if } p \leq 1. \]

6. We studied Taylor series and showed that some can be found using the formula for convergent geometric series. Two examples of Taylor series that were obtained in this way are
\[ \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad \text{for } |x| < 1 \]
and
\[ \arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \quad \text{for } |x| < 1 \]

7. In discussing Taylor series, we considered some of the following questions: (a) For what range of values of \( x \) can we expect the series to converge? (b) Suppose we approximate the function on the right by a finite number of terms on the left. How good is that approximation? Another interesting question is: (c) If we include more and more such terms, does that approximation get better and better? (i.e., does the series converge to the function?) (d) Is the convergence rate rapid? Some of these questions occupy the attention of career mathematicians, and are beyond the scope of this introductory calculus course.

8. More generally, we showed that the Taylor series for a function about \( x = 0 \),
\[ f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_{k=0}^{\infty} a_k x^k. \]
can be found by computing the coefficients
\[ a_k = \frac{f^{(k)}(0)}{k!} \]

9. We discussed some of the applications of Taylor series. We used Taylor series to approximate a function, to find an approximation for a definite integral of a function, and to solve a differential equation.