Chapter 6

Techniques of Integration

In this chapter, we expand our repertoire for antiderivatives beyond the “elementary” functions discussed so far. A review of the table of elementary antiderivatives (found in Chapter 3) will be useful. Here we will discuss a number of methods for finding antiderivatives. We refer to these collected “tricks” as methods of integration. As will be shown, in some cases, these methods are systematic (i.e. with clear steps), whereas in other cases, guesswork and trial and error is an important part of the process.

A primary method of integration to be described is substitution. A close relationship exists between the chain rule of differential calculus and the substitution method. A second very important method is integration by parts. Aside from its usefulness in integration per se, this method has numerous applications in physics, mathematics, and other sciences. Many other techniques of integration used to form a core of methods taught in such courses in integral calculus. Many of these are quite technical. Nowadays, with sophisticated mathematical software packages (including Maple and Mathematica), integration can be carried out automatically via computation called “symbolic manipulation”, reducing our need to dwell on these technical methods.

6.1 Differential notation

We begin by familiarizing the reader with notation that appears frequently in substitution integrals, i.e. differential notation. Consider a straight line

\[ y = mx + b. \]

Recall that the slope of the line, \( m \), is

\[ m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}. \]

This relationship can also be written in the form

\[ \Delta y = m \Delta x. \]
Figure 6.1. The slope of the line shown here is \( m = \frac{\Delta y}{\Delta x} \). This means that the small quantities \( \Delta y \) and \( \Delta x \) are related by \( \Delta y = m \Delta x \).

If we take a very small step along this line in the \( x \) direction, call it \( dx \) (to remind us of an "infinitesimally small" quantity), then the resulting change in the \( y \) direction, (call it \( dy \)) is related by

\[
dy = m \, dx.
\]

Now suppose that we have a curve defined by some arbitrary function, \( y = f(x) \) which need not be a straight line. For a given point \((x, y)\) on this curve, a step \( \Delta x \) in the \( x \) direction is associated with a step \( \Delta y \) in the \( y \) direction. The relationship between the step sizes is:

\[
\Delta y = m_s \Delta x,
\]

where now \( m_s \) is the slope of a secant line (shown connecting two points on the curve in Figure 6.2). If the sizes of the steps are small (\( dx \) and \( dy \)), then this relationship is well approximated by the slope of the tangent line, \( m_t \) as shown in Figure 6.2 i.e.

\[
dy = m_t \, dx = f'(x) \, dx.
\]
The quantities $dx$ and $dy$ are called *differentials*. In general, they link a small step on the $x$ axis with the resulting small change in height along the tangent line to the curve (shown in Figure 6.2). We might observe that the ratio of the differentials, i.e.

$$\frac{dy}{dx} = f'(x),$$

appears to link our result to the definition of the derivative. We remember, though, that the derivative is actually defined as a limit:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

When the step size $\Delta x$ is quite small, it is approximately true that

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}.$$

This notation will be useful in substitution integrals.

**Examples**

We give some examples of functions, their derivatives, and the differential notation that goes with them.

1. The function $y = f(x) = x^3$ has derivative $f'(x) = 3x^2$. Thus

$$dy = 3x^2 \, dx.$$

2. The function $y = f(x) = \tan(x)$ has derivative $f'(x) = \sec^2(x)$. Therefore

$$dy = \sec^2(x) \, dx.$$

3. The function $y = f(x) = \ln(x)$ has derivative $f'(x) = \frac{1}{x}$ so

$$dy = \frac{1}{x} \, dx.$$

With some practice, we can omit the intermediate step of writing down a derivative and go directly from function to differential notation. Given a function $y = f(x)$ we will often write

$$df(x) = \frac{df}{dx} \, dx$$

and occasionally, we use just the symbol $df$ to mean the same thing. The following examples illustrate this idea with specific functions.

$$d(\sin(x)) = \cos(x) \, dx, \quad d(x^n) = nx^{n-1} \, dx, \quad d(\arctan(x)) = \frac{1}{1 + x^2} \, dx.$$

Moreover, some of the basic rules of differentiation translate directly into rules for handling and manipulating differentials. We give a list of some of these elementary rules below.
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Rules for derivatives and differentials

1. \( \frac{d}{dx} C = 0, \quad dC = 0 \).

2. \( \frac{d}{dx} (u(x) + v(x)) = \frac{du}{dx} + \frac{dv}{dx} \quad d(u + v) = du + dv \).

3. \( \frac{d}{dx} u(x)v(x) = \frac{du}{dx}v + u\frac{dv}{dx} \quad d(uv) = u \, dv + v \, du \).

4. \( \frac{d}{dx} (Cu(x)) = C \frac{du}{dx}, \quad d(Cu) = C \, du \).

6.2 Antidifferentiation and indefinite integrals

In Chapters 2 and 3, we defined the concept of the definite integral, which represents a number. It will be useful here to consider the idea of an indefinite integral, which is a function, namely an antiderivative.

If two functions, \( F(x) \) and \( G(x) \), have the same derivative, say \( f(x) \), then they differ at most by a constant, that is \( F(x) = G(x) + C \), where \( C \) is some constant.

Proof

Since \( F(x) \) and \( G(x) \) have the same derivative, we have

\[
\frac{d}{dx} F(x) = \frac{d}{dx} G(x),
\]

\[
\frac{d}{dx} F(x) - \frac{d}{dx} G(x) = 0,
\]

\[
\frac{d}{dx} (F(x) - G(x)) = 0.
\]

This means that the function \( F(x) - G(x) \) should be a constant, since its derivative is zero. Thus

\[
F(x) - G(x) = C,
\]

so

\[
F(x) = G(x) + C,
\]

as required. \( F(x) \) and \( G(x) \) are called antiderivatives of \( f(x) \), and this confirms, once more, that any two antiderivatives differ at most by a constant.

In another terminology, which means the same thing, we also say that \( F(x) \) (or \( G(x) \)) is the integral of the function \( f(x) \), and we refer to \( f(x) \) as the integrand. We write this as follows:

\[
F(x) = \int f(x) \, dx.
\]

This notation is sometimes called “an indefinite integral” because it does not denote a specific numerical value, nor is an interval specified for the integration range. An indefinite
integral is a function with an arbitrary constant. (Contrast this with the definite integral studied in our last chapters: in the case of the definite integral, we specified an interval, and interpreted the result, a number, in terms of areas associated with curves.) We also write

$$\int f(x) \, dx = F(x) + C,$$

if we want to indicate the form of all possible functions that are antiderivatives of $f(x)$. $C$ is referred to as a constant of integration.

### 6.2.1 Integrals of derivatives

Suppose we are given an integral of the form

$$\int \frac{df}{dx} \, dx,$$

or alternately, the same thing written using differential notation,

$$\int df.$$

How do we handle this? We reason as follows. The $df/dx$ (a quantity that is, itself, a function) is the derivative of the function $f(x)$. That means that $f(x)$ is the antiderivative of $df/dx$. Then, according to the Fundamental Theorem of Calculus,

$$\int \frac{df}{dx} \, dx = f(x) + C.$$

We can write this same result using the differential of $f$, as follows:

$$\int df = f(x) + C.$$

The following examples illustrate the idea with several elementary functions.

### Examples

1. $\int d\cos x = \cos x + C$.
2. $\int dv = v + C$.
3. $\int dx^3 = x^3 + C$.

### 6.3 Simple substitution

In this section, we observe that the forms of some integrals can be simplified by making a judicious substitution, and using our familiarity with derivatives (and the chain rule). The idea rests on the fact that in some cases, we can spot a “helper function”

$$u = f(x),$$
such that the quantity
\[ du = f'(x)dx \]
appears in the integrand. In that case, the substitution will lead to eliminating \( x \) entirely in favour of the new quantity \( u \), and simplification may occur.

### 6.3.1 Example: Simple substitution

Suppose we are given the function

\[ f(x) = (x + 1)^{10}. \]

Then its antiderivative (indefinite integral) is

\[ F(x) = \int f(x) \, dx = \int (x + 1)^{10} \, dx. \]

We could find an antiderivative by expanding the integrand \((x + 1)^{10}\) into a degree 10 polynomial and using methods already known to us; but this would be laborious. Let us observe, however, that if we define

\[ u = (x + 1), \]

then

\[ du = \frac{d(x + 1)}{dx} \, dx = \left( \frac{dx}{dx} + \frac{d(1)}{dx} \right) \, dx = (1 + 0) \, dx = dx. \]

Now replacing \((x + 1)\) by \(u\) and \(dx\) by the equivalent \(du\) we get:

\[ F(x) = \int u^{10} \, du. \]

An antiderivative to this can be easily found, namely,

\[ F(x) = \frac{u^{11}}{11} = \frac{(x + 1)^{11}}{11} + C. \]

In the last step, we converted the result back to the original variable, and included the arbitrary integration constant. A very important point to remember is that we can always check our results by differentiation:

**Check**

Differentiate \( F(x) \) to obtain

\[ \frac{dF}{dx} = \frac{1}{11} (11(x + 1)^{10}) = (x + 1)^{10}. \]
6.3.2 How to handle endpoints

We consider how substitution type integrals can be calculated when we have endpoints, i.e. in evaluating definite integrals. Consider the example:

\[ I = \int_1^2 \frac{1}{x+1} \, dx. \]

This integration can be done by making the substitution \( u = x + 1 \) for which \( du = dx \). We can handle the endpoints in one of two ways:

**Method 1: Change the endpoints**

We can change the integral over entirely to a definite integral in the variable \( u \) as follows: Since \( u = x + 1 \), the endpoint \( x = 1 \) corresponds to \( u = 2 \), and the endpoint \( x = 2 \) corresponds to \( u = 3 \), so changing the endpoints to reflect the change of variables leads to

\[ I = \left[ \frac{1}{u} \ln |u| \right]_2^3 = \ln 3 - \ln 2 = \ln \frac{3}{2}. \]

In the last steps we have plugged in the new endpoints (appropriate to \( u \)).

**Method 2: Change back to \( x \) before evaluating at endpoints**

Alternately, we could rewrite the antiderivative in terms of \( x \).

\[ \int \frac{1}{u} \, du = \ln |u| = \ln |x + 1| \]

and then evaluate this function at the original endpoints.

\[ \left. \int_1^2 \frac{1}{x+1} \, dx = \ln |x+1| \right|_1^2 = \ln \frac{3}{2} \]

Here we plugged in the original endpoints (as appropriate to the variable \( x \)).

6.3.3 Examples: Substitution type integrals

Find a simple substitution and determine the antiderivatives (indefinite or definite integrals) of the following functions:

1. \[ I = \int \frac{2}{x+2} \, dx. \]
2. \[ I = \int_0^1 x^2 e^{x^3} \, dx \]
3. \[ I = \int \frac{1}{(x+1)^2 + 1} \, dx. \]
4. \( I = \int (x + 3) \sqrt{x^2 + 6x + 10} \, dx \).

5. \( I = \int_0^\pi \cos^3(x) \sin(x) \, dx \).

6. \( I = \int \frac{1}{ax + b} \, dx \).

7. \( I = \int \frac{1}{b + ax^2} \, dx \).

Solutions

1. \( I = \int \frac{2}{x + 2} \, dx \). Let \( u = x + 2 \). Then \( du = dx \) and we get
   \[ I = \int \frac{2}{u} \, du = 2 \int \frac{1}{u} \, du = 2 \ln |u| = 2 \ln |x + 2| + C. \]

2. \( I = \int_0^1 x^2 e^{x^3} \, dx \). Let \( u = x^3 \). Then \( du = 3x^2 \, dx \). Here we use method 2 for handling endpoints.
   \[ \int e^u \, du = 3 \int e^u \, du = \frac{1}{3} e^u + C. \]
   Then
   \[ I = \int_0^1 x^2 e^{x^3} \, dx = \frac{1}{3} e^1 \bigg|_0^1 = \frac{1}{3} (e - 1). \]
   (We converted the antiderivative to the original variable, \( x \), before plugging in the original endpoints.)

3. \( I = \int \frac{1}{(x + 1)^2 + 1} \, dx \). Let \( u = x + 1 \), then \( du = dx \) so we have
   \[ I = \int \frac{1}{u^2 + 1} \, du = \arctan(u) = \arctan(x + 1) + C. \]

4. \( I = \int (x + 3) \sqrt{x^2 + 6x + 10} \, dx \). Let \( u = x^2 + 6x + 10 \). Then \( du = (2x + 6) \, dx = 2(x + 3) \, dx \). With this substitution we get
   \[ I = \int \sqrt{u} \, \frac{du}{2} = \frac{1}{2} \int u^{1/2} \, du = \frac{1}{2} \frac{u^{3/2}}{3/2} = \frac{1}{3} u^{3/2} = \frac{1}{3} (x^2 + 6x + 10)^{3/2} + C. \]

5. \( I = \int_0^\pi \cos^3(x) \sin(x) \, dx \). Let \( u = \cos(x) \). Then \( du = -\sin(x) \, dx \). Here we use method 1 for handling endpoints. For \( x = 0, u = \cos 0 = 1 \) and for \( x = \pi, u = \cos \pi = -1 \), so changing the integral and endpoints to \( u \) leads to
   \[ I = \int_1^{-1} u^3 (-du) = -\frac{u^4}{4} \bigg|_1^{-1} = -\frac{1}{4}((-1)^4 - 1^4) = 0. \]
6.3. Simple substitution

Here we plugged in the new endpoints that are relevant to the variable $u$

1. $\int \frac{1}{ax + b} \, dx$. Let $u = ax + b$. Then $du = a \, dx$, so $dx = du/a$. Substitute the above equations into the first equation and simplify to get

\[
I = \int \frac{1}{u} \, du = \frac{1}{a} \int \frac{1}{u} \, du = \frac{1}{a} \ln |u| + C.
\]

Substitute $u = ax + b$ back to arrive at the solution

\[
I = \int \frac{1}{ax + b} \, dx = \frac{1}{a} \ln |ax + b| \tag{6.1}
\]

2. $I = \int \frac{1}{b + ax^2} \, dx$. This can be brought to the form of an arctan type integral as follows: Let $u^2 = (a/b)x^2$, so $u = \sqrt{a/b} x$ and $du = \sqrt{a/b} \, dx$. Now substituting these, we get

\[
I = \frac{1}{b} \int \frac{1}{1 + u^2} \, du = \frac{1}{b} \arctan \left( \frac{u}{\sqrt{a/b}} \right) = \frac{1}{\sqrt{ba}} \arctan \left( \sqrt{a/b} \, x \right) + C.
\]

6.3.4 When simple substitution fails

Not every integral can be handled by simple substitution. Let us see what could go wrong:

**Example: Substitution that does not work**

Consider the case

\[
F(x) = \int \sqrt{1 + x^2} \, dx = \int (1 + x^2)^{1/2} \, dx.
\]

A “reasonable” guess for substitution might be

\[
u = (1 + x^2).
\]

Then

\[
du = 2x \, dx,
\]

and $dx = du/2x$. Attempting to convert the integral to the form containing $u$ would lead to

\[
I = \int \sqrt{u} \, du.
\]

We have not succeeded in eliminating $x$ entirely, so the expression obtained contains a mixture of two variables. We can proceed no further. This substitution did not simplify the integral and we must try some other technique.
6.3.5 Checking your answer

Finding an antiderivative can be tricky. (To a large extent, methods described in this chapter are a “collection of tricks”.) However, it is always possible (and wise) to check for correctness, by differentiating the result. This can help uncover errors.

For example, suppose that (in the previous example) we had incorrectly guessed that the antiderivative of
\[ \int (1 + x^2)^{1/2} \, dx \]
might be
\[ F_{\text{guess}}(x) = \frac{1}{3/2}(1 + x^2)^{3/2}. \]
The following check demonstrates the incorrectness of this guess: Differentiate \( F_{\text{guess}}(x) \) to obtain
\[ F'_{\text{guess}}(x) = \frac{1}{3/2}(3/2)(1 + x^2)^{(3/2) - 1} \cdot 2x = (1 + x^2)^{1/2} \cdot 2x. \]
The result is clearly not the same as \((1 + x^2)^{1/2}\), since an “extra” factor of \(2x\) appears from application of the chain rule: this means that the trial function \( F_{\text{guess}}(x) \) was not the correct antiderivative. (We can similarly check to confirm correctness of any antiderivative found by following steps of methods here described. This can help to uncover sign errors and other algebraic mistakes.)

6.4 More substitutions

In some cases, rearrangement is needed before the form of an integral becomes apparent. We give some examples in this section. The idea is to reduce each one to the form of an elementary integral, whose antiderivative is known.

<table>
<thead>
<tr>
<th>Standard integral forms</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( I = \int \frac{1}{u} , du = \ln</td>
</tr>
<tr>
<td>2. ( I = \int u^n , du = \frac{u^{n+1}}{n+1}. )</td>
</tr>
<tr>
<td>3. ( I = \int \frac{1}{1 + u^2} , du = \arctan(u) + C. )</td>
</tr>
</tbody>
</table>

However, finding which of these forms is appropriate in a given case will take some ingenuity and algebra skills. Integration tends to be more of an art than differentiation, and recognition of patterns plays an important role here.

6.4.1 Example: perfect square in denominator

Find the antiderivative for
\[ I = \int \frac{1}{x^2 - 6x + 9} \, dx. \]
6.4. More substitutions

Solution

We observe that the denominator of the integrand is a perfect square, i.e. that \( x^2 - 6x + 9 = (x - 3)^2 \). Replacing this in the integral, we obtain

\[
I = \int \frac{1}{x^2 - 6x + 9} \, dx = \int \frac{1}{(x - 3)^2} \, dx.
\]

Now making the substitution \( u = (x - 3) \), and \( du = dx \) leads to a power type integral

\[
I = \int \frac{1}{u^2} \, du = \int u^{-2} \, du = -u^{-1} = -\frac{1}{(x - 3)} + C.
\]

6.4.2 Example: completing the square

A small change in the denominator will change the character of the integral, as shown by this example:

\[
I = \int \frac{1}{x^2 - 6x + 10} \, dx.
\]

Solution

Here we use “completing the square” to express the denominator in the form \( x^2 - 6x + 10 = (x - 3)^2 + 1 \). Then the integral takes the form

\[
I = \int \frac{1}{1 + (x - 3)^2} \, dx.
\]

Now a substitution \( u = (x - 3) \) and \( du = dx \) will result in

\[
I = \int \frac{1}{1 + u^2} \, du = \arctan(u) = \arctan(x - 3) + C.
\]

Remark: in cases where completing the square gives rise to a constant other than 1 in the denominator, we use the technique illustrated in Example 6.3.3 Eqn. (6.1) to simplify the problem.

6.4.3 Example: factoring the denominator

A change in one sign can also lead to a drastic change in the antiderivative. Consider

\[
I = \int \frac{1}{1 - x^2} \, dx.
\]

In this case, we can factor the denominator to obtain

\[
I = \int \frac{1}{(1 - x)(1 + x)} \, dx.
\]
We will show shortly that the integrand can be simplified to the sum of two fractions, i.e. that

\[ I = \int \frac{1}{(1-x)(1+x)} \, dx = \int \frac{A}{1-x} + \frac{B}{1+x} \, dx, \]

where \( A, B \) are constants. The algebraic technique for finding these constants, and hence of forming the simpler expressions, called Partial fractions, will be discussed in an upcoming section. Once these constants are found, each of the resulting integrals can be handled by substitution.

### 6.5 Trigonometric substitutions

Trigonometric functions provide a rich set of interconnected functions that show up in many problems. It is useful to remember three very important trigonometric identities that help to simplify many integrals. These are:

#### Essential trigonometric identities

1. \( \sin^2(x) + \cos^2(x) = 1 \)
2. \( \sin(A + B) = \sin(A) \cos(B) + \sin(B) \cos(A) \)
3. \( \cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B) \).

In the special case that \( A = B = x \), the last two identities above lead to:

#### Double angle trigonometric identities

1. \( \sin(2x) = 2 \sin(x) \cos(x) \).
2. \( \cos(2x) = \cos^2(x) - \sin^2(x) \).

From these, we can generate a variety of other identities as special cases. We list the most useful below. The first two are obtained by combining the double-angle formula for cosines with the identity \( \sin^2(x) + \cos^2(x) = 1 \).

#### Useful trigonometric identities

1. \( \cos^2(x) = \frac{1 + \cos(2x)}{2} \).
2. \( \sin^2(x) = \frac{1 - \cos(2x)}{2} \).
3. \( \tan^2(x) + 1 = \sec^2(x) \).

### 6.5.1 Example: simple trigonometric substitution

Find the antiderivative of

\[ I = \int \sin(x) \cos^2(x) \, dx. \]
6.5. Trigonometric substitutions

Solution

This integral can be computed by a simple substitution, similar to Example 5 of Section 6.3. We let \( u = \cos(x) \) and \( du = -\sin(x) \, dx \) to get the integral into the form

\[
I = - \int u^2 \, du = -\frac{u^3}{3} = -\frac{\cos^3(x)}{3} + C.
\]

We need none of the trigonometric identities in this case. Simple substitution is always the easiest method to use. It should be the first method attempted in each case.

6.5.2 Example: using trigonometric identities (1)

Find the antiderivative of

\[
I = \int \cos^2(x) \, dx.
\]

Solution

This is an example in which the “Useful trigonometric identity” 1 leads to a simpler integral. We write

\[
I = \int \cos^2(x) \, dx = \int \frac{1 + \cos(2x)}{2} \, dx = \frac{1}{2} \int (1 + \cos(2x)) \, dx.
\]

Then clearly,

\[
I = \frac{1}{2} \left( x + \frac{\sin(2x)}{2} \right) + C.
\]

6.5.3 Example: using trigonometric identities (2)

Find the antiderivative of

\[
I = \int \sin^3(x) \, dx.
\]

Solution

We can rewrite this integral in the form

\[
I = \int \sin^2(x) \sin(x) \, dx.
\]

Now using the trigonometric identity \( \sin^2(x) + \cos^2(x) = 1 \), leads to

\[
I = \int (1 - \cos^2(x)) \sin(x) \, dx.
\]

This can be split up into

\[
I = \int \sin(x) \, dx - \int \sin(x) \cos^2(x) \, dx.
\]
The first part is elementary, and the second was shown in a previous example. Therefore we end up with

\[ I = -\cos(x) + \frac{\cos^3(x)}{3} + C. \]

Note that it is customary to combine all constants obtained in the calculation into a single constant, \( C \) at the end.

Aside from integrals that, themselves, contain trigonometric functions, there are other cases in which use of trigonometric identities, though at first seemingly unrelated, is crucial. Many expressions involving the form \( \sqrt{1 \pm x^2} \) or the related form \( \sqrt{a \pm bx^2} \) will be simplified eventually by conversion to trigonometric expressions!

### 6.5.4 Example: converting to trigonometric functions

Find the antiderivative of

\[ I = \int \sqrt{1 - x^2} \, dx. \]

**Solution**

The simple substitution \( u = 1 - x^2 \) will not work, (as shown by a similar example in Section 6.3). However, converting to trigonometric expressions will do the trick. Let

\[ x = \sin(u), \quad \text{then} \quad dx = \cos(u) \, du. \]

(In Figure 6.3, we show this relationship on a triangle. This diagram is useful in reversing the substitutions after the integration step.) Then \( 1 - x^2 = 1 - \sin^2(u) = \cos^2(u) \), so the substitutions lead to

\[ I = \int \sqrt{\cos^2(u)} \cos(u) \, du = \int \cos^2(u) \, du. \]

From a previous example, we already know how to handle this integral. We find that

\[ I = \frac{1}{2} \left( u + \frac{\sin(2u)}{2} \right) = \frac{1}{2} (u + \sin(u) \cos(u)) + C. \]
6.5. Trigonometric substitutions

(In the last step, we have used the double angle trigonometric identity. We will shortly see why this simplification is relevant.)

We now desire to convert the result back to a function of the original variable, \( x \).

We note that \( x = \sin(u) \) implies \( u = \arcsin(x) \). To convert the term \( \cos(u) \) back to an expression depending on \( x \) we can use the relationship \( 1 - \sin^2(u) = \cos^2(u) \), to deduce that

\[
\cos(u) = \sqrt{1 - \sin^2(u)} = \sqrt{1 - x^2}.
\]

It is sometimes helpful to use a Pythagorean triangle, as shown in Figure 6.3, to rewrite the antiderivative in terms of the variable \( x \). The idea is this: We construct the triangle in such a way that its side lengths are related to the “angle” \( u \) according to the substitution rule. In this example, \( x = \sin(u) \) so the sides labeled \( x \) and 1 were chosen so that their ratio (“opposite over hypotenuse” coincides with the sine of the indicated angle, \( u \), thereby satisfying \( x = \sin(u) \). We can then determine the length of the third leg of the triangle (using the Pythagorean formula) and thus all other trigonometric functions of \( u \). For example, we note that the ratio of “adjacent over hypotenuse” is \( \cos(u) = \sqrt{1 - x^2}/1 = \sqrt{1 - x^2} \). Finally, with these reverse substitutions, we find that,

\[
I = \int \sqrt{1 - x^2} \, dx = \frac{1}{2} \left( \arcsin(x) + x \sqrt{1 - x^2} \right) + C.
\]

**Remark:** In computing a definite integral of the same type, we can circumvent the need for the conversion back to an expression involving \( x \) by using the appropriate method for handling endpoints. For example, the integral

\[
I = \int_0^1 \sqrt{1 - x^2} \, dx
\]

can be transformed to

\[
I = \int_0^{\pi/2} \sqrt{\cos^2(u) \cos(u)} \, du,
\]

by observing that \( x = \sin(u) \) implies that \( u = 0 \) when \( x = 0 \) and \( u = \pi/2 \) when \( x = 1 \). Then this means that the integral can be evaluated directly (without changing back to the variable \( x \)) as follows:

\[
I = \int_0^{\pi/2} \sqrt{\cos^2(u) \cos(u)} \, du = \frac{1}{2} \left( u + \sin(2u) \right) \bigg|_0^{\pi/2} = \frac{1}{2} \left( \frac{\pi}{2} + \sin(\pi) \right) = \frac{\pi}{4}
\]

where we have used the fact that \( \sin(\pi) = 0 \).

Some subtle points about the domains of definition of inverse trigonometric functions will not be discussed here in detail. (See material on these functions in a first term calculus course.) Suffice it to say that some integrals of this type will be undefined if this endpoint conversion cannot be carried out (e.g. if the interval of integration had been \( 0 \leq x \leq 2 \), we would encounter an impossible relation \( 2 = \sin(u) \). Since no value of \( u \) satisfies this relation, such a definite integral has no meaning, i.e. “does not exist”.)
6.5.5 Example: The centroid of a two dimensional shape

We extend the concept of centroid (center of mass) for a region that has uniform density in 2D, but where we consider the distribution of mass along the \( x \) (or \( y \)) axis. Consider the semicircle shape of uniform thickness, shown in Figure 6.4, and suppose it is balanced along its horizontal edge. Find the \( x \) coordinate \( \bar{c} \) at which the shape balances.

\[
y = \sqrt{9 - x^2}
\]

**Figure 6.4. A semicircular shape.**

**Solution**

The semicircle is one quarter of a circle of radius 3. Its edge is described by the equation

\[
y = f(x) = \sqrt{9 - x^2}.
\]

We will assume that the density per unit area is uniform. However, the mass per unit length along the \( x \) axis is not uniform, due to the shape of the object. We apply the idea of integration: If we cut the shape at increments of \( \Delta x \) along the \( x \) axis, we get a collection of pieces whose mass is each proportional to \( f(x) \Delta x \). Summing up such contributions and letting the widths \( \Delta x \to dx \) get small, we arrive at the integral for mass. The total mass of the shape is thus

\[
M = \int_{0}^{3} f(x) \, dx = \int_{0}^{3} \sqrt{9 - x^2} \, dx.
\]

Furthermore, if we compute the integral

\[
I = \int_{0}^{3} x f(x) \, dx = \int_{0}^{3} x \sqrt{9 - x^2} \, dx,
\]

we obtain the \( x \) coordinate of the center of mass,

\[
\bar{x} = \frac{I}{M}.
\]

It is evident that the mass is proportional to the area of one quarter of a circle of radius 3:

\[
M = \frac{1}{4} \pi (3)^2 = \frac{9}{4} \pi.
\]
(We could also see this by performing a trigonometric substitution integral.) The second integral can be done by simple substitution. Consider

\[ I = \int_0^3 x f(x) \, dx = \int_0^3 x \sqrt{9 - x^2} \, dx. \]

Let \( u = 9 - x^2 \). Then \( du = -2x \, dx \). The endpoints are converted as follows: \( x = 0 \Rightarrow u = 9 - 0^2 = 9 \) and \( x = 3 \Rightarrow u = 9 - 3^2 = 0 \) so that we get the integral

\[ I = \int_9^0 \sqrt{u} \frac{1}{-2} \, du. \]

We can reverse the endpoints if we switch the sign, and this leads to

\[ I = \frac{1}{2} \int_0^9 u^{1/2} \, du = \left( \frac{1}{2} \right) \left( \frac{u^{3/2}}{3/2} \right) \bigg|_0^9. \]

Since \( 9^{3/2} = (9^{1/2})^3 = 3^3 \), we get \( I = (3^3)/3 = 3^2 = 9 \). Thus the \( x \) coordinate of the center of mass is

\[ \bar{x} = \frac{I}{M} = \frac{9}{(9/4)\pi} = \frac{4}{\pi}. \]

We can similarly find the \( y \) coordinate of the center of mass: To do so, we would express the boundary of the shape in the form \( x = f(y) \) and integrate to find

\[ \bar{y} = \int_0^3 y f(y) \, dy. \]

For the semicircle, \( y^2 + x^2 = 9 \), so \( x = f(y) = \sqrt{9 - y^2} \). Thus

\[ \bar{y} = \int_0^3 y \sqrt{9 - y^2} \, dy. \]

This integral looks identical to the one we wrote down for \( \bar{x} \). Thus, based on this similarity (or based on the symmetry of the problem) we will find that

\[ \bar{y} = \frac{4}{\pi}. \]

### 6.5.6 Example: tan and sec substitution

Find the antiderivative of

\[ I = \int \sqrt{1 + x^2} \, dx. \]

**Solution**

We aim for simplification by the identity \( 1 + \tan^2(u) = \sec^2(u) \), so we set

\[ x = \tan(u), \quad dx = \sec^2(u) \, du. \]
Then the substitution leads to

\[ I = \int \sqrt{1 + \tan^2(u)} \sec^2(u) \, du = \int \sqrt{\sec^2(u)} \sec^2(u) \, du = \int \sec^3(u) \, du. \]

This integral will require further work, and will be partly calculated by Integration by Parts in Appendix 11.5. In this example, the triangle shown in Figure 6.5 shows the relationship between \( x \) and \( u \) and will help to convert other trigonometric functions of \( u \) to functions of \( x \).

![Figure 6.5. As in Figure 6.3 but for example 6.5.6.](image)

### 6.6 Partial fractions

In this section, we show a simple algebraic trick that helps to simplify an integrand when it is in the form of some rational function such as

\[ f(x) = \frac{1}{(ax + b)(cx + d)}. \]

The idea is to break this up into simpler rational expressions by finding constants \( A, B \) such that

\[
\frac{1}{(ax + b)(cx + d)} = \frac{A}{ax + b} + \frac{B}{cx + d}.
\]

Each part can then be handled by a simple substitution, as shown in Example 6.3.3, Eqn. (6.1). We give several examples below.

#### 6.6.1 Example: partial fractions (1)

Find the antiderivative of

\[ I = \int \frac{1}{x^2 - 1}. \]

Factoring the denominator, \( x^2 - 1 = (x - 1)(x + 1) \), suggests breaking up the integrand into the form

\[
\frac{1}{x^2 - 1} = \frac{A}{x + 1} + \frac{B}{x - 1}.
\]
The two sides are equal provided:
\[
\frac{1}{x^2 - 1} = \frac{A(x - 1) + B(x + 1)}{x^2 - 1}.
\]

This means that
\[
1 = A(x - 1) + B(x + 1)
\]

must be true for all \(x\) values. We now ask what values of \(A\) and \(B\) make this equation hold for any \(x\). Choosing two “easy” values, namely \(x = 1\) and \(x = -1\) leads to isolating one or the other unknown constants, \(A, B\), with the results:
\[
1 = -2A, \quad 1 = 2B.
\]

Thus \(B = 1/2, A = -1/2\), so the integral can be written in the simpler form
\[
I = \frac{1}{2} \left( \int \frac{-1}{(x + 1)} \, dx + \int \frac{1}{(x - 1)} \, dx \right).
\]

(A common factor of \((1/2)\) has been taken out.) Now a simple substitution will work for each component. (Let \(u = x + 1\) for the first, and \(u = x - 1\) for the second integral.) The result is
\[
I = \int \frac{1}{x^2 - 1} = \frac{1}{2} (-\ln |x + 1| + \ln |x - 1|) + C.
\]

### 6.6.2 Example: partial fractions (2)

Find the antiderivative of
\[
I = \int \frac{1}{x(1 - x)} \, dx.
\]

This example is similar to the previous one. We set
\[
\frac{1}{x(1 - x)} = \frac{A}{x} + \frac{B}{1 - x}.
\]

Then
\[
1 = A(1 - x) + Bx.
\]

This must hold for all \(x\) values. In particular, convenient values of \(x\) for determining the constants are \(x = 0, 1\). We find that
\[
A = 1, \quad B = 1.
\]

Thus
\[
I = \int \frac{1}{x(1 - x)} \, dx = \int \frac{1}{x} \, dx + \int \frac{1}{1 - x} \, dx.
\]

Simple substitution now gives
\[
I = \ln |x| - \ln |1 - x| + C.
\]
6.6.3 Example: partial fractions (3)

Find the antiderivative of

\[ I = \int \frac{x}{x^2 + x - 2}. \]

The rational expression above factors into \( x^2 + x - 2 = (x - 1)(x + 2) \), leading to the expression

\[ \frac{x}{x^2 + x - 2} = \frac{A}{x - 1} + \frac{B}{x + 2}. \]

Consequently, it follows that

\[ A(x + 2) + B(x - 1) = x. \]

Substituting the values \( x = 1, -2 \) into this leads to \( A = 1/3 \) and \( B = 2/3 \). The usual procedure then results in

\[ I = \int \frac{x}{x^2 + x - 2} = \frac{1}{3} \ln |x - 1| + \frac{2}{3} \ln |x + 2| + C. \]

Another example of the technique of partial fractions is provided in Appendix 11.5.2.

6.7 Integration by parts

The method described in this section is important as an additional tool for integration. It also has independent theoretical stature in many applications in mathematics and physics. The essential idea is that in some cases, we can exchange the task of integrating a function with the job of differentiating it.

The idea rests on the product rule for derivatives. Suppose that \( u(x) \) and \( v(x) \) are two differentiable functions. Then we know that the derivative of their product is

\[ \frac{d(uv)}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}, \]

or, in the differential notation:

\[ d(uv) = v \, du + u \, dv, \]

Integrating both sides, we obtain

\[ \int d(uv) = \int v \, du + \int u \, dv \]

i.e.

\[ uv = \int v \, du + \int u \, dv. \]

We write this result in the more suggestive form

\[ \int u \, dv = uv - \int v \, du. \]

The idea here is that if we have difficulty evaluating an integral such as \( \int u \, dv \), we may be able to “exchange it” for a simpler integral in the form \( \int v \, du \). This is best illustrated by the examples below.
Example: Integration by parts (1)
Compute
\[ I = \int_{1}^{2} \ln(x) \, dx. \]

Solution
Let \( u = \ln(x) \) and \( dv = dx \). Then \( du = \frac{1}{x} \, dx \) and \( v = x \).
\[ \int \ln(x) \, dx = x \ln(x) - \int x \left( \frac{1}{x} \right) \, dx = x \ln(x) - \int dx = x \ln(x) - x. \]
We now evaluate this result at the endpoints to obtain
\[ I = \int_{1}^{2} \ln(x) \, dx = \left. \left( x \ln(x) - x \right) \right|_{1}^{2} = (2 \ln(2) - 2) - (1 \ln(1) - 1) = 2 \ln(2) - 1. \]
(Where we used the fact that \( \ln(1) = 0 \).)

Example: Integration by parts (2)
Compute
\[ I = \int_{0}^{1} xe^x \, dx. \]

Solution
At first, it may be hard to decide how to assign roles for \( u \) and \( dv \). Suppose we try \( u = e^x \) and \( dv = x \, dx \). Then \( du = e^x \, dx \) and \( v = \frac{x^2}{2} \). This means that we would get the integral in the form
\[ I = \frac{x^2}{2} e^x - \int \frac{x^2}{2} e^x \, dx. \]
This is certainly not a simplification, because the integral we obtain has a higher power of \( x \), and is consequently harder, not easier to integrate. This suggests that our first attempt was not a helpful one. (Note that integration often requires trial and error.)

Let \( u = x \) and \( dv = e^x \, dx \). This is a wiser choice because when we differentiate \( u \), we reduce the power of \( x \) (from 1 to 0), and get a simpler expression. Indeed, \( du = dx \), \( v = e^x \) so that
\[ \int xe^x \, dx = xe^x - \int e^x \, dx = xe^x - e^x + C. \]
To find a definite integral of this kind on some interval (say \( 0 \leq x \leq 1 \)), we compute
\[ I = \int_{0}^{1} xe^x \, dx = \left. \left( xe^x - e^x \right) \right|_{0}^{1} = (1e^1 - e^1) - (0e^0 - e^0) = 0 + e^0 = e^0 = 1. \]
Note that all parts of the expression are evaluated at the two endpoints.
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Example: Integration by parts (2b)

Compute

$$I_n = \int x^n e^x \, dx.$$  

Solution

We can calculate this integral by repeated application of the idea in the previous example. Letting $u = x^n$ and $dv = e^x \, dx$ leads to $du = nx^{n-1}$ and $v = e^x$. Then

$$I_n = x^n e^x - \int nx^{n-1} e^x \, dx.$$  

Each application of integration by parts reduces the power of the term $x^n$ inside an integral by one. The calculation is repeated until the very last integral has been simplified, with no remaining powers of $x$. This illustrates that in some problems, integration by parts is needed more than once.

Example: Integration by parts (3)

Compute

$$I = \int \arctan(x) \, dx.$$  

Solution

Let $u = \arctan(x)$ and $dv = dx$. Then $du = (1/(1 + x^2)) \, dx$ and $v = x$ so that

$$I = x \arctan(x) - \int \frac{1}{1 + x^2} x \, dx.$$  

The last integral can be done with the simple substitution $w = (1 + x^2)$ and $dw = 2x \, dx$, giving

$$I = x \arctan(x) - (1/2) \int (1/w) \, dw.$$  

We obtain, as a result

$$I = x \arctan(x) - \frac{1}{2} \ln(1 + x^2).$$  

Example: Integration by parts (3b)

Compute

$$I = \int \tan(x) \, dx.$$  

6.7. Integration by parts

Solution

We might try to fit this into a similar pattern, i.e. let \( u = \tan(x) \) and \( dv = dx \). Then \( du = \sec^2(x) \, dx \) and \( v = x \), so we obtain

\[
I = x \tan(x) - \int x \sec^2(x) \, dx.
\]

This is not really a simplification, and we see that integration by parts will not necessarily work, even on a seemingly related example. However, we might instead try to rewrite the integral in the form

\[
I = \int \tan(x) \, dx = \int \frac{\sin(x)}{\cos(x)} \, dx.
\]

Now we find that a simple substitution will do the trick, i.e. that \( w = \cos(x) \) and \( dw = -\sin(x) \, dx \) will convert the integral into the form

\[
I = \int \frac{1}{w} (-dw) = -\ln |w| = -\ln |\cos(x)|.
\]

This example illustrates that we should always try substitution, first, before attempting other methods.

Example: Integration by parts (4)

Compute

\[
I_1 = \int e^x \sin(x) \, dx.
\]

We refer to this integral as \( I_1 \) because a related second integral, that we’ll call \( I_2 \) will appear in the calculation.

Solution

Let \( u = e^x \) and \( dv = \sin(x) \, dx \). Then \( du = e^x \, dx \) and \( v = -\cos(x) \, dx \). Therefore

\[
I_1 = -e^x \cos(x) - \int (-\cos(x))e^x \, dx = -e^x \cos(x) + \int \cos(x)e^x \, dx.
\]

We now have another integral of a similar form to tackle. This seems hopeless, as we have not simplified the result, but let us not give up! In this case, another application of integration by parts will do the trick. Call \( I_2 \) the integral

\[
I_2 = \int \cos(x)e^x \, dx,
\]

so that

\[
I_1 = -e^x \cos(x) + I_2.
\]

Repeat the same procedure for the new integral \( I_2 \), i.e. Let \( u = e^x \) and \( dv = \cos(x) \, dx \). Then \( du = e^x \, dx \) and \( v = \sin(x) \, dx \). Thus

\[
I_2 = e^x \sin(x) - \int \sin(x)e^x \, dx = e^x \sin(x) - I_1.
\]
This appears to be a circular argument, but in fact, it has a purpose. We have determined that the following relationships are satisfied by the above two integrals:

\[ I_1 = -e^x \cos(x) + I_2 \]
\[ I_2 = e^x \sin(x) - I_1. \]

We can eliminate \( I_2 \), obtaining

\[ I_1 = -e^x \cos(x) + I_2 = -e^x \cos(x) + e^x \sin(x) - I_1. \]

that is,

\[ I_1 = -e^x \cos(x) + e^x \sin(x) - I_1. \]

Rearranging (taking \( I_1 \) to the left hand side) leads to

\[ 2I_1 = -e^x \cos(x) + e^x \sin(x) \]

and thus, the desired integral has been found to be

\[ I_1 = \int e^x \sin(x) \, dx = \frac{1}{2} (-e^x \cos(x) + e^x \sin(x)) = \frac{1}{2} e^x (\sin(x) - \cos(x)) + C. \]

(At this last step, we have included the constant of integration.) Moreover, we have also found that \( I_2 \) is related, i.e. using \( I_2 = e^x \sin(x) - I_1 \) we now know that

\[ I_2 = \int \cos(x)e^x \, dx = \frac{1}{2} e^x (\sin(x) + \cos(x)) + C. \]

### 6.8 Summary

In this chapter, we explored a number of techniques for computing antiderivatives. We here summarize the most important results:

1. Substitution is the first method to consider. This method works provided the change of variable results in elimination of the original variable and leads to a simpler, more elementary integral.

2. When using substitution on a definite integral, endpoints can be converted to the new variable (Method 1) or the resulting antiderivative can be converted back to its original variable before plugging in the (original) endpoints (Method 2).

3. The integration by parts formula for functions \( u(x), v(x) \) is

\[ \int u \, dv = uv - \int v \, du. \]

Integration by parts is useful when \( u \) is easy to differentiate (but not easy to integrate). It is also helpful when the integral contains a product of elementary functions such as \( x^n \) and a trigonometric or an exponential function. Sometimes more than one application of this method is needed. Other times, this method is combined with substitution or other simplifications.
4. Using integration by parts on a definite integral means that both parts of the formula are to be evaluated at the endpoints.

5. Integrals involving $\sqrt{1 + x^2}$ can be simplified by making a trigonometric substitution.

6. Integrals with products or powers of trigonometric functions can sometimes be simplified by application of trigonometric identities or simple substitution.

7. Algebraic tricks, and many associated manipulations are often applied to twist and turn a complicated integral into a set of simpler expressions that can each be handled more easily.

8. Even with all these techniques, the problem of finding an antiderivative can be very complicated. In some cases, we resort to handbooks of integrals, use symbolic manipulation software packages, or, if none of these work, calculate a given definite integral numerically using a spreadsheet.

**Table of elementary antiderivatives**

1. $\int \frac{1}{u} \, du = \ln |u| + C.$
2. $\int u^n \, du = \frac{u^{n+1}}{n+1} + C$
3. $\int \frac{1}{1 + u^2} \, du = \arctan(u) + C$
4. $\int \frac{1}{\sqrt{1 - x^2}} \, du = \arcsin(u) + C$
5. $\int \sin(u) \, du = -\cos(u) + C$
6. $\int \cos(u) \, du = \sin(u) + C$
7. $\int \sec^2(u) \, du = \tan(u) + C$

**Additional useful antiderivatives**

1. $\int \tan(u) \, du = \ln |\sec(u)| + C.$
2. $\int \cot(u) \, du = \ln |\sin(u)| + C$
3. $\int \sec(u) \, du = \ln |\sec(u) + \tan(u)| + C$