Chapter 9
Differential Equations

9.1 Introduction

A differential equation is a relationship between some (unknown) function and one of its derivatives. Examples of differential equations were encountered in an earlier calculus course in the context of population growth, temperature of a cooling object, and speed of a moving object subjected to friction. In Section 4.2.4, we reviewed an example of a differential equation for velocity, (4.8), and discussed its solution, but here, we present a more systematic approach to solving such equations using a technique called separation of variables. In this chapter, we apply the tools of integration to finding solutions to differential equations. The importance and wide applicability of this topic cannot be overstated.

In this course, since we are concerned only with functions that depend on a single variable, we discuss ordinary differential equations (ODE’s), whereas later, after a multivariate calculus course where partial derivatives are introduced, a wider class, of partial differential equations (PDE’s) can be studied. Such equations are encountered in many areas of science, and in any quantitative analysis of systems where rates of change are linked to the state of the system. Most laws of physics are of this form; for example, applying the familiar Newton’s law, \( F = ma \), links the position of a pendulum’s mass to its acceleration (second derivative of position).\(^{42}\) Many biological processes are also described by differential equations. The rate of growth of a population \( \frac{dN}{dt} \) depends on the size of that population at the given time \( N(t) \).

Constructing the differential equation that adequately represents a system of interest is an art that takes some thought and experience. In this process, which we call “modeling”, many simplifications are made so that the essential properties of a given system are captured, leaving out many complicating details. For example, friction might be neglected in “modeling” a perfect pendulum. The details of age distribution might be neglected in modeling a growing population. Now that we have techniques for integration, we can devise a new approach to computing solutions of differential equations.

Given a differential equation and a starting value, the goal is to make a prediction

\(^{42}\)Newton’s law states that force is proportional to acceleration. For a pendulum, the force is due to gravity, and the acceleration is a second derivative of the x or y coordinate of the bob on the pendulum.
about the future behaviour of the system. This is equivalent to identifying the function that satisfies the given differential equation and initial value(s). We refer to such a function as the solution to the initial value problem (IVP). In differential calculus, our exploration of differential equations was limited to those whose solution could be guessed, or whose solution was supplied in advance. We also explored some of the fascinating geometric and qualitative properties of such equations and their predictions.

Now that we have techniques of integration, we can find the analytic solution to a variety of simple first-order differential equations (i.e. those involving the first derivative of the unknown function). We will describe the technique of separation of variables. This technique works for examples that are simple enough that we can isolate the dependent variable (e.g. \( y \)) on one side of the equation, and the independent variable (e.g. time \( t \)) on the other side.

### 9.2 Unlimited population growth

We start with a simple example that was treated thoroughly in the differential calculus semester of this course. We consider a population with per capita birth and mortality rates that are constant, irrespective of age, disease, environmental changes, or other effects. We ask how a population in such ideal circumstances would change over time. We build up a simple model (i.e. a differential equation) to describe this ideal case, and then proceed to find its solution. Solving the differential equation is accomplished by a new technique introduced here, namely separation of variables. This reduces the problem to integration and algebraic manipulation, allowing us to compute the population size at any time \( t \). By going through this process, we essentially convert information about the rate of change and starting level of the population to a detailed prediction of the population at later times.43

#### 9.2.1 A simple model for population growth

Let \( y(t) \) represent the size of a population at time \( t \). We will assume that at time \( t = 0 \), the population level is specified, i.e. \( y(0) = y_0 \) is some given constant. We want to find the population at later times, given information about birth and mortality rates, (both of which are here assumed to be constant over time).

The population changes through births and mortality. Suppose that \( b > 0 \) is the per capita average birth rate, and \( m > 0 \) the per capita average mortality rate. The assumption that \( b, m \) are both constants is a simplification that neglects many biological effects, but will be used for simplicity in this first example.

The statement that the population increases through births and decreases due to mortality, can be restated as

\[
\text{rate of change of } y = \text{rate of births} - \text{rate of mortality}
\]

where the rate of births is given by the product of the per capita average birth rate \( b \) and the population size \( y \). Similarly, the rate of mortality is given by \( my \). Translating the rate of

\[43\text{Of course, we must keep in mind that such predictions are based on simplifying assumptions, and are to be taken as an approximation of any real population growth.}\]
9.2. Unlimited population growth

Change into the corresponding derivative of $y$ leads to

$$\frac{dy}{dt} = by - my = (b - m)y.$$  

Let us define the new constant,

$$k = b - m.$$  

Then $k$ is the net per capita growth rate of the population. We can distinguish two possible cases: $b > m$ means that there are more births then deaths, so we expect the population to grow. $b < m$ means that there are more deaths than births, so that the population will eventually go extinct. There is also a marginal case that $b = m$, for which $k = 0$, where the population does not change at all. To summarize, this simple model of unlimited growth leads to the differential equation and initial condition:

$$\frac{dy}{dt} = ky, \quad y(0) = y_0. \tag{9.1}$$

Recall that a differential equation together with an initial condition is called an initial value problem. To find a solution to such a problem, we look for the function $y(t)$ that describes the population size at any future time $t$, given its initial size at time $t = 0$.

9.2.2 Separation of variables and integration

We here introduce the technique, separation of variables, that will be used in all the examples described in this chapter. Since the differential equation (9.1) is relatively simple, this first example will be relatively straightforward. We would like to determine $y(t)$ given the differential equation

$$\frac{dy}{dt} = ky.$$  

Rather than integrating this equation as is\footnote{We may be tempted to integrate both sides of this equation with respect to the independent variable $t$, e.g. writing $\int \frac{dy}{dt} \, dt = \int ky \, dt + C$, (where $C$ is some constant), but this is not very useful, since the integral on the right hand side (RHS) can only be carried out if we know the function $y = y(t)$, which we are trying to determine.}, we use an alternate approach, considering $dt$ and $dy$ as “differentials” in the sense defined in Section 6.1. We rearrange and rewrite the above equation in the form

$$\frac{1}{y} \, dy = k \, dt, \tag{9.2}$$

This step of putting expressions involving the independent variable $t$ on one side and expressions involving the dependent variable $y$ on the opposite side gives rise to the name “separation of variables”.

Now, the LHS of Eqn. (9.2) depends only on the variable $y$, and the RHS only on $t$. The constant $k$ will not interfere with any integration step. Moreover, integrating each side of Eqn. (9.2) can be carried out independently.

To determine the appropriate intervals for integration, we observe that when time sweeps over some interval $0 \leq t \leq T$ (from initial to final time), the value of $y(t)$ will
change over a corresponding interval $y_0 \leq y \leq y(T)$. Here $y_0$ is the given starting value of $y$ (prescribed by the initial condition in (9.1)). We do not yet know $y(T)$, but our goal is to find that value, i.e. to predict the future behaviour of $y$. Integrating leads to

$$\int_{y_0}^{y(T)} \frac{1}{y} \, dy = \int_0^T k \, dt = k \int_0^T dt,$$

$$\ln |y| \bigg|_{y_0}^{y(T)} = kt \bigg|_0^T,$$

$$\ln |y(T)| - \ln |y(0)| = k(T - 0),$$

$$\ln \left| \frac{y(T)}{y_0} \right| = kT,$$

$$\frac{y(T)}{y_0} = e^{kT},$$

$$y(T) = y_0 e^{kT}.$$  

But this result holds for any arbitrary final time, $T$. In other words, since this is true for any time we chose, we can set $T = t$, arriving at the desired solution

$$y(t) = y_0 e^{kt}. \tag{9.3}$$

The above formula relates the predicted value of $y$ at any time $t$ to its initial value, and to all the parameters of the problem. Observe that plugging in $t = 0$, we get $y(0) = y_0 e^{k0} = y_0 e^0 = y_0$, so that the solution (9.3) satisfies the initial condition. We leave as an exercise for the reader\textsuperscript{45} to validate that the function in (9.3) also satisfies the differential equation in (9.1).

By solving the initial value problem (9.1), we have determined that, under ideal conditions, when the net per capita growth rate $t$ is constant, a population will grow exponentially with time. Recall that this validates results that we had encountered in our first calculus course.

\section*{9.3 Terminal velocity and steady states}

Here we revisit the equation for velocity of a falling object that we first encountered in Section 4.2.4. We wish to derive the appropriate differential equation governing that velocity, and find the solution $v(t)$ as a function of time. We will first reconsider the simplest case of uniformly accelerated motion (i.e. where friction is neglected), as in Section 4.2.3. We then include friction, as in Section 4.2.4 and use the new technique of separation of variables to shortcut the method of solution.

\textsuperscript{45}This kind of check is good practice and helps to spot errors. Simply differentiate Eqn. (9.3) and show that the result is the same as $k$ times the original function, as required by the equation (9.1).
9.3. Terminal velocity and steady states

9.3.1 Ignoring friction: the uniformly accelerated case

Let $v(t)$ and $a(t)$ be the velocity and the acceleration, respectively, of an object falling under the force of gravity at time $t$. We take the positive direction to be downwards, for convenience. Suppose that at time $t = 0$, the object starts from rest, i.e. the initial velocity of the object is known to be $v(0) = 0$. When friction is neglected, the object will accelerate,

$$a(t) = g,$$

which is equivalent to the statement that the velocity increases at a constant rate,

$$\frac{dv}{dt} = g. \quad (9.4)$$

Because $g$ is constant, we do not need to use separation of variables, i.e. we can integrate each side of this equation directly. Writing

$$\int \frac{dv}{dt} \, dt = \int g \, dt + C = g \int dt + C,$$

where $C$ is an integration constant, we arrive at

$$v(t) = gt + C. \quad (9.5)$$

Here we have used (on the LHS) that $v$ is the antiderivative of $dv/dt$. (equivalently, we can simplify the integral $\int \frac{dv}{dt} \, dt = \int dv = v$). Plugging in $v(0) = 0$ into Eqn. (9.5) leads to $0 = g \cdot 0 + C = C$, so the constant we need is $C = 0$ and the velocity satisfies

$$v(t) = gt.$$  

We have just arrived at a result that parallels Eqn. (4.4) of Section 4.2.3 (in slightly different notation).

9.3.2 Including friction: the case of terminal velocity

When a falling object experiences the force of friction, it cannot accelerate indefinitely. In fact, a frictional force retards the downwards motion. To a good approximation, that force is proportional to the velocity.

A force balance for the falling object leads to

$$ma(t) = mg - \gamma v(t),$$

where $\gamma$ is the frictional coefficient. For an object of constant mass, we can divide through by $m$, so

$$a(t) = g - \frac{\gamma}{m} v(t).$$

It is important to note the distinction between this simple example and other cases where separation of variables is required. It would not be wrong to use separation of variables to find the solution for Eqn. (9.4), but it would just be “overkill”, since simple integration of the each side of the equation “as is” does the job.
Let \( k = \gamma/m \). Then, the velocity at any time satisfies the differential equation and initial condition

\[
\frac{dv}{dt} = g - kv, \quad v(0) = 0.
\]  

We can find the solution to this differential equation and predict the velocity at any time \( t \) using separation of variables.

Figure 9.1. The velocity \( v(t) \) as a function of time given by Eqn. (9.7) as found in Section 9.3.2. Note that as time increases, the velocity approaches some constant terminal velocity. The parameters used were \( g = 9.8 \text{ m/s}^2 \) and \( k = 0.5 \).

Consider a time interval \( 0 \leq t \leq T \), and suppose that, during this time interval, the velocity changes from an initial value of \( v(0) = 0 \) to the final value, \( v(T) \) at the final time, \( T \). Then using separation of variables and integration, we get

\[
\frac{dv}{dt} = g - kv,
\]

\[
\frac{dv}{g - kv} = dt,
\]

\[
\int_0^{v(T)} \frac{dv}{g - kv} = \int_0^T dt.
\]
Substitute \( u = g - kv \) for the integral on the left hand side. Then \( du = -kdv, dv = (-1/k)du \), so we get an integral of the form

\[
-\frac{1}{k} \int \frac{1}{u} \, du = -\frac{1}{k} \ln |u|.
\]

After replacing \( u \) by \( g - kv \), we arrive at

\[
-\frac{1}{k} \ln |g - kv| \bigg|_0^T = t |
\]

We use the fact that \( v(0) = 0 \) to write this as

\[
-\frac{1}{k} \left( \ln |g - kv(T)| - \ln |g| \right) = T,
\]

\[
-\frac{1}{k} \left( \ln \frac{|g - kv(T)|}{g} \right) = T,
\]

\[
\ln \frac{|g - kv(T)|}{g} = -kT.
\]

We are finished with the integration step, but the function we are trying to find, \( v(T) \), is still tangled up inside an expression involving the natural logarithm. Extricating it will involve some subtle reasoning about signs because there is an absolute value to contend with. As a first step, we exponentiate both sides to remove the logarithm.

\[
\left| \frac{g - kv(T)}{g} \right| = e^{-kT} \Rightarrow |g - kv(T)| = ge^{-kT}.
\]

Because the constant \( g \) is positive, we could remove absolute values signs from it. To simplify further, we have to consider the sign of the term inside the absolute value in the numerator. In the case we are considering here, \( v(0) = 0 \). This will mean that the quantity \( g - kv(T) \) is always be non-negative (i.e. \( g - kv(T) \geq 0 \)). We will verify this fact shortly. For the moment, supposing this is true, we can write

\[
|g - kv(T)| = g - kv(T) = ge^{-kT},
\]

and finally solve for \( v(T) \) to obtain our final result,

\[
v(T) = \frac{g}{k}(1 - e^{-kT}).
\]

Here we note that \( v(T) \) can never be larger than \( g/k \) since the term \( (1 - e^{-kT}) \) is always \( \leq 1 \). Hence, we were correct in assuming that \( g - kv(T) \geq 0 \).

As before, the above formula relating velocity to time holds for any choice of the final time \( T \), so we can write, in general,

\[
v(t) = \frac{g}{k}(1 - e^{-kt}). \quad (9.7)
\]
This is the solution to the initial value problem (9.6). It predicts the velocity of the falling object through time. Note that we have arrived once more at the result obtained in Eqn. (4.11), but using the technique of separation of variables\footnote{It often happens that a differential equation can be solved using several different methods.}

We graph the expression given in (9.7) in Figure 9.1. Note that as $t$ increases, the term $e^{-kt}$ decreases rapidly, so that the velocity approaches a constant whose value is

$$v(t) \to \frac{g}{k}.$$  

We call this the terminal velocity\footnote{A similar plot of the solution of the differential equation (9.6) could be assembled using Euler’s method, as studied in differential calculus. That is the numerical method alternative to the analytic technique discussed in this chapter. The student may wish to review results obtained in a previous semester to appreciate the correspondence.}.

### 9.3.3 Steady state

We might observe that the terminal velocity can also be found quite simply and directly from the differential equation itself. It is the steady state of the differential equation, i.e. the value for which no further change takes place. The steady state can be found by setting the derivative in the differential equation, to zero, i.e. by letting

$$\frac{dv}{dt} = 0.$$  

When this is done, we arrive at

$$g - kv = 0 \quad \Rightarrow \quad v = \frac{g}{k}.$$  

Thus, at steady state, the velocity of the falling object is indeed the same as the terminal velocity that we have just discovered.

### 9.4 Related problems and examples

The example discussed in Section 9.3.2 belongs to a class of problems that share many common features. Generally, this class is represented by linear differential equations of the form

$$\frac{dy}{dt} = a - by,$$  

(9.8)  

with given initial condition $y(0) = y_0$. Properties of this equation were studied in the context of differential calculus in a previous semester. Now, with the same method as we applied to the problem of terminal velocity, we can integrate this equation by separation of variables, writing

$$\frac{dy}{a - by} = dt$$  

and proceeding as in the previous example. We arrive at its solution,

$$y(t) = \frac{a}{b} + \left( y_0 - \frac{a}{b} \right) e^{-bt}.$$  

(9.9)
The steps are left as an exercise for the reader.

We observe that the steady state of the above equation is obtained by setting

\[
\frac{dy}{dt} = a - by = 0, \quad \text{i.e.} \quad y = \frac{a}{b}.
\]

Indeed the solution given in the formula (9.9) has the property that as \( t \) increases, the exponential term \( e^{-bt} \to 0 \) so that the term in large brackets will vanish and \( y \to a/b \). This means that from any initial value, \( y \) will approach its steady state level.

This equation has a number of important applications that arise in a variety of context. A few of these are mentioned below.

### 9.4.1 Blood alcohol

Let \( y(t) \) be the level of alcohol in the blood of an individual during a party. Suppose that the average rate of drinking is gradual and constant (i.e. small sips are continually taken, so that the rate of input of alcohol is approximately constant). Further, assume that alcohol is detoxified in the liver at a rate proportional to its blood level. Then an equation of the form (9.8) would describe the blood level over the period of drinking. \( y(0) = 0 \) would signify the absence of alcohol in the body at the beginning of the evening. The constant \( a \) would reflect the rate of intake per unit volume of the individual's blood: larger people take longer to "get drunk" for a given amount consumed\(^{49}\). The constant \( b \) represents the rate of decay of alcohol per unit time due to degradation by the liver, assumed constant\(^{50}\); young healthy drinkers have a higher value of \( b \) than those who can no longer metabolize alcohol as efficiently.

The solution (9.9) has several features of note: it illustrates the fact that alcohol would increase from the initial level, but only up to a maximum of \( a/b \), where the intake and degradation balance. Indeed, the level \( y = a/b \) represents a steady state level (as long as drinking continues). Of course, this level could be toxic to the drinker, and the assumptions of the model may break down in that region! In the phase of "recovery", after drinking stops, the above differential equation no longer describes the level of blood alcohol. Instead, the process of recovery is represented by

\[
\frac{dy}{dt} = -by, \quad y(0) = y_0. \tag{9.10}
\]

The level of blood alcohol then decays exponentially with rate \( b \) from its level at the moment that drinking ends. We show this typical pattern in Figure 9.2.

### 9.4.2 Chemical kinetics

The same ideas apply to any chemical substance that is formed at a constant rate (or supplied at a constant rate) \( a \), and then breaks down with rate proportional to its concentration. We then call the constant \( b \) the "decay rate constant".

\(^{49}\)Of course, we are here assuming a constant intake rate, as though the alcohol is being continually sipped all evening at a uniform rate. Most people do not drink this way, instead quaffing a few large drinks over some hours. It is possible to describe this, but we will not do so in this chapter.

\(^{50}\)This is also a simplifying assumption, as the rate of metabolism can depend on other factors, such as food intake.
Figure 9.2. The level of alcohol in the blood is described by Eqn. (9.8) for the first two hours of drinking. At $t = 2h$, the drinking stopped (so $a = 0$ from then on). The level of alcohol in the blood then decays back to zero, following Eqn. (9.10).

The variable $y(t)$ represents the concentration of chemical at time $t$, and the same differential equation describes this chemical process. As above, given any initial level of the substance, $y(t) = y_0$, the level of $y$ will eventually approach the steady state, $y = a/b$.

9.5 Emptying a container

In this section we investigate a new problem in which the differential equation that describes a process will be derived from basic physical principles\(^{51}\). We will look at the flow of fluid leaking out of a container, and use mass balance to derive a differential equation model. When this is done, we will also use separation of variables to predict how long it takes for the container to be emptied.

We will assume that the container has a small hole at its base. The rate of emptying of the container will depend on the height of fluid in the container above the hole\(^{52}\). We can derive a simple differential equation that describes the rate that the height of the fluid changes using the following physical argument.

9.5.1 Conservation of mass

Suppose that the container is a cylinder, with a constant cross sectional area $A > 0$, as shown in Fig. 9.3. Suppose that the area of the hole is $a$. The rate that fluid leaves through the hole must balance with the rate that fluid decreases in the container. This principle is called mass balance. We will here assume that the density of water is constant, so that we can talk about the net changes in volume (rather than mass).

\(^{51}\)This example is particularly instructive. First, it shows precisely how physical laws can be combined to formulate a model, then it shows how the problem can be recast as a single ODE in one dependent variable. Finally, it illustrates a slightly different integral.\n
\(^{52}\)As we have assumed that the hole is at $h = 0$, we henceforth consider the height of the fluid surface, $h(t)$ to be the same as "the height of fluid above the hole".
9.5. Emptying a container

We investigate the time it takes to empty a container full of fluid by deriving a differential equation model and solving it using the methods developed in this chapter. $A$ is the cross-sectional area of the cylindrical tank, $a$ is the cross-sectional area of the hole through which fluid drains, $v(t)$ is the velocity of the fluid, and $h(t)$ is the time dependent height of fluid remaining in the tank (indicated by the dashed line). The volume of fluid leaking out in a time span $\Delta t$ is $av \Delta t$ - see small cylindrical volume indicated on the right.

We refer to $V(t)$ as the volume of fluid in the container at time $t$. Note that for the cylindrical container, $V(t) = Ah(t)$ where $A$ is the cross-sectional area and $h(t)$ is the height of the fluid at time $t$. The rate of change of $V$ is

$$\frac{dV}{dt} = -(\text{rate volume lost as fluid flows out}).$$

(The minus sign indicates that the volume is decreasing).

At every second, some amount of fluid leaves through the hole. Suppose we are told that the velocity of the water molecules leaving the hole is precisely $v(t)$ in units of cm/sec. (We will find out how to determine this velocity shortly.) Then in one second, those particles have moved a distance $v$ cm/sec $\cdot$ 1 sec $= v$ cm. In fact, all the particles in a little cylinder of length $v$ behind these molecules have also left the hole. Indeed, if we know the area of the hole, we can determine precisely what volume of water exits through the hole each second, namely

$$\text{rate volume lost as fluid flows out} = va.$$ 

(The small inset in Fig. 9.3 shows a little “cylindrical unit” of fluid that flows out of the hole per second. The area is $a$ and the length of that little volume is $v$. Thus the volume leaving per second is $va$.)

So far we have a relationship between the volume of fluid in the tank and the velocity of the water exiting the hole:

$$\frac{dV}{dt} = -av.$$ 

Now we need to determine the velocity $v$ of the flow to complete the formulation of the problem.
9.5.2 Conservation of energy

The fluid “picks up speed” because it has “dropped” by a height \( h \) from the top of the fluid surface to the hole. In doing so, a small mass of water has simply exchanged some potential energy (due to its relative height above the hole) for kinetic energy (expressed by how fast it is moving). Potential energy of a small mass of water (\( m \)) at height \( h \) will be \( mgh \), whereas when the water flows out of the hole, its kinetic energy is given by \( (1/2)mv^2 \) where \( v \) is velocity. Thus, for these to balance (so that total energy is conserved) we have

\[
\frac{1}{2}mv^2 = mgh.
\]

(Here \( v = v(t) \) is the instantaneous velocity of the fluid leaving the hole and \( h = h(t) \) is the height of the water column.) This allows us to relate the velocity of the fluid leaving the hole to the height of the water in the tank, i.e.

\[
v^2 = 2gh \quad \Rightarrow \quad v = \sqrt{2gh}.
\]  

(9.11)

In fact, both the height of fluid and its exit velocity are constantly changing as the fluid drains, so we might write \([v(t)]^2 = 2gh(t)\) or \( v(t) = \sqrt{2gh(t)} \). We have arrived at this result using an energy balance argument.

9.5.3 Putting it together

We now combine the various pieces of information to arrive at the model, a differential equation for a single (unknown) function of time. There are three time-dependent variables that were discussed above, the volume \( V(t) \), the height \( h(t) \), of the velocity \( v(t) \). It proves convenient to express everything in terms of the height of water in the tank, \( h(t) \), though this choice is to some extent arbitrary. Keeping units in an equation consistent is essential. Checking for unit consistency can help to uncover errors in equations, including differential equations.

Recall that the volume of the water in the tank, \( V(t) \) is related to the height of fluid \( h(t) \) by

\[
V(t) = Ah(t),
\]

where \( A > 0 \) is a constant, the cross-sectional area of the tank. We can simplify as follows:

\[
\frac{dV}{dt} = \frac{d(Ah(t))}{dt} = A\frac{d(h(t))}{dt}.
\]

But by previous steps and Eqn. (9.11)

\[
\frac{dV}{dt} = -av = -a\sqrt{2gh}.
\]

Thus

\[
A\frac{d(h(t))}{dt} = -a\sqrt{2gh},
\]

or simply put,

\[
\frac{dh}{dt} = -\frac{a}{A}\sqrt{2gh} = -k\sqrt{h}.
\]  

(9.12)
where $k$ is a constant that depends on the size and shape of the cylinder and its hole:

$$k = \frac{a}{A} \sqrt{2g}.$$

If the area of the hole is very small relative to the cross-sectional area of the tank, then $k$ will be very small, so that the tank will drain very slowly (i.e. the rate of change in $h$ per unit time will not be large). On a planet with a very high gravitational force, the same tank will drain more quickly. A taller column of water drains faster. Once its height has been reduced, its rate of draining also slows down. We comment that Equation (9.12) has a minus sign, signifying that the height of the fluid decreases.

Using simple principles such as conservation of mass and conservation of energy, we have shown that the height $h(t)$ of water in the tank at time $t$ satisfies the differential equation (9.12). Putting this together with the initial condition (height of fluid $h_0$ at time $t = 0$), we arrive at initial value problem to solve:

$$\frac{dh}{dt} = -k\sqrt{h}, \quad h(0) = h_0.$$  \hspace{1cm} (9.13)

Clearly, this equation is valid only for $h$ non-negative. We also remark that Eqn. (9.13) is nonlinear as it involves the variable $h$ in a nonlinear term, $\sqrt{h}$. Next, we use separation of variables to find the height as a function of time.

### 9.5.4 Solution by separation of variables

The equation (9.13) shows how height of fluid is related to its rate of change, but we are interested in an explicit formula for fluid height $h$ versus time $t$. To obtain that relationship, we must determine the solution to this differential equation. We do this using separation of variables. (We will also use the initial condition $h(0) = h_0$ that accompanies Eqn. (9.13).)

As usual, rewrite the equation in the separated form,

$$\frac{dh}{\sqrt{h}} = -kdt.$$

We integrate from $t = 0$ to $t = T$, during which the height of fluid that started as $h_0$ becomes some new height $h(T)$ to be determined.

$$\int_{h_0}^{h(T)} \frac{1}{\sqrt{h}} dh = -k \int_0^T dt.$$

Now integrate both sides and simplify:

$$\frac{h^{1/2}}{(1/2)} \bigg|_{h_0}^{h(T)} = -kT$$

$$2 \left( \sqrt{h(T)} - \sqrt{h_0} \right) = -kT$$

\footnote{In many cases, nonlinear differential equations are more challenging than linear ones. However, examples chosen in this chapter are simple enough that we will not experience the true challenges of such nonlinearities.}
\[ \sqrt{h(T)} = -k \frac{T}{2} + \sqrt{h_0} \]

\[ h(T) = \left( \sqrt{h_0} - k \frac{T}{2} \right)^2. \]

Since this is true for any time \( t \), we can also write the form of the solution as

\[ h(t) = \left( \sqrt{h_0} - k \frac{t}{2} \right)^2. \] (9.14)

Eqn. (9.14) predicts fluid height remaining in the tank versus time \( t \). In Fig. 9.4 we show some of the “solution curves”\(^{54}\), i.e. functions of the form Eqn. (9.14) for a variety of initial fluid height values \( h_0 \). We can also use our results to predict the emptying time, as shown in the next section.

---

54 As before, this figure was produced by plotting the analytic solution (9.14). A numerical method alternative would use Euler’s Method and the spreadsheet to obtain the (approximate) solution directly from the initial value problem (9.13).
9.6. Density dependent growth

9.5.5 How long will it take the tank to empty?

The tank will be empty when the height of fluid is zero. Setting \( h(t) = 0 \) in Eqn. 9.14

\[
\left( \sqrt{h_0} - k \frac{t^2}{2} \right)^2 = 0.
\]

Solving this equation for the emptying time \( t_e \), we get

\[
k \frac{t_e^2}{2} = \sqrt{h_0} \Rightarrow t_e = \frac{2 \sqrt{h_0}}{k}.
\]

The time it takes to empty the tank depends on the initial height of water in the tank. Three examples are shown in Figure 9.4 for initial heights of \( h_0 = 2.5, 5, 10 \). The emptying time depends on the square-root of the initial height. This means, for instance, that doubling the height of fluid initially in the tank only increases the time it takes by a factor of \( \sqrt{2} \approx 1.41 \).

Making the hole smaller has a more direct “proportional” effect, since we have found that \( k = (a/A) \sqrt{2g} \).

9.6 Density dependent growth

The simple model discussed in Section 9.2 for population growth has an unrealistic feature of unlimited explosive exponential growth. To correct for this unrealistic feature, a common assumption is that the rate of growth is “density dependent”. In this section, we consider a revised differential equation that describes such growth, and use the new tools to analyze its predictions. In place of our previous notation we will now use \( N \) to represent the size of a population.

9.6.1 The logistic equation

The logistic equation is the simplest density dependent growth equation, and we study its behaviour below.

Let \( N(t) \) be the size of a population at time \( t \). Clearly, we expect \( N(t) \geq 0 \) for all time \( t \), since a population cannot be negative. We will assume that the initial population is known, \( N(0) = N_0 \). The logistic differential equation states that the rate of change of the population is given by

\[
\frac{dN}{dt} = rN \left( \frac{K - N}{K} \right).
\]

Here \( r > 0 \) is called the intrinsic growth rate and \( K > 0 \) is called the carrying capacity. \( K \) reflects that size of the population that can be sustained by the given environment. We can understand this equation as a modified growth law in which the “density dependent” term, \( r(K - N)/K \), replaces the previous constant net growth rate \( k \).
9.6.2 Scaling the equation

The form of the equation can be simplified if we measure the population in units of the carrying capacity, instead of “numbers of individuals”. i.e. if we define a new quantity

\[ y(t) = \frac{N(t)}{K}. \]

This procedure is called scaling. To see this, consider dividing each side of the logistic equation (9.15) by the constant \( K \). Then

\[ \frac{1}{K} \frac{dN}{dt} = \frac{r}{K} \left( \frac{K - N}{K} \right). \]

We now group terms conveniently, forming

\[ d \left( \frac{N}{K} \right) = \frac{r}{K} \left( \frac{N}{K} \right) \left( 1 - \left( \frac{N}{K} \right) \right). \]

Replacing \( (N/K) \) by \( y \) in each case, we obtain the scaled equation and initial condition given by

\[ \frac{dy}{dt} = ry(1 - y), \quad y(0) = y_0. \quad (9.16) \]

Now the variable \( y(t) \) measures population size in “units” of the carrying capacity, and \( y_0 = N_0/K \) is the scaled initial population level. Here again is an initial value problem, like Eqn. (9.13), but unlike Eqn. (9.1), the logistic differential equation is nonlinear. That is, the variable \( y \) appears in a nonlinear expression (in fact a quadratic) in the equation.

9.6.3 Separation of variables

Here we will solve Eqn. (9.16) by separation of variables. The idea is essentially the same as our previous examples, but is somewhat more involved. To show an alternative method of handling the integration, we will treat both sides as indefinite integrals. Separating the variables leads to

\[ \int \frac{1}{y(1 - y)} \, dy = r \int dt \]

\[ \int \frac{1}{y(1 - y)} \, dy = \int r \, dt + K. \]

The integral on the right will lead to \( rt + K \) where \( K \) is some constant of integration that we need to incorporate since we do not have endpoints on our integrals. But we must work harder to evaluate the integral on the left. We can do so by partial fractions, the technique described in Section 6.6. Details are given in Section 9.6.4.

9.6.4 Application of partial fractions

Let

\[ I = \int \frac{1}{y(1 - y)} \, dy. \]
Then for some constants $A, B$ we can write

$$I = \int \frac{A}{y} + \frac{B}{1-y} \, dy = A \ln |y| - B \ln |1 - y|.$$  

(The minus sign in front of $B$ stems from the fact that letting $u = 1 - y$ would lead to $du = -dy$.) We can find $A, B$ from the fact that

$$\frac{A}{y} + \frac{B}{1-y} = \frac{1}{y(1-y)},$$

so that

$$A(1-y) + By = 1.$$  

This must be true for all $y$, and in particular, substituting in $y = 0$ and $y = 1$ leads to $A = 1, B = 1$ so that

$$I = \ln |y| - \ln |1 - y| = \ln \left| \frac{y}{1-y} \right|.$$  

### 9.6.5 The solution of the logistic equation

We now have to extract the quantity $y$ from the equation

$$\left( \ln \left| \frac{y}{1-y} \right| \right) = rt + K.$$  

That is, we want $y$ as a function of $t$. After exponentiating both sides we need to remove the absolute value. We will now assume that $y$ is initially smaller than 1, and show that it remains so. In that case, everything inside the absolute value is positive, and we can write

$$\frac{y(t)}{(1 - y(t))} = e^{rt+K} = e^K e^{rt} = C e^{rt}.$$  

In the above step, we have simply renamed the constant, $e^K$ by the new name $C$ for simplicity. $C > 0$ is now also an arbitrary constant whose value will be determined from the initial conditions. Indeed, if we substitute $t = 0$ into the most recent equation, we find that

$$\frac{y(0)}{(1 - y(0))} = Ce^0 = C,$$

so that

$$C = \frac{y_0}{(1 - y_0)}.$$  

We will use this fact shortly. What remains now is some algebra to isolate the desired function $y(t)$

$$y(t) = (1 - y(t))Ce^{rt},$$

$$y(t) \left( 1 + Ce^{rt} \right) = Ce^{rt}. $$
The desired function is now expressed in terms of the time $t$, and the constants $r, C$. We can also express it in terms of the initial value of $y$, i.e. $y_0$, by using what we know to be true about the constant $C$, i.e. $C = y_0/(1 - y_0)$. When we do so, we arrive at

$$ y(t) = \frac{1}{1 + y_0 e^{-rt} + 1} = \frac{y_0}{y_0 + (1 - y_0)e^{-rt}}. \quad (9.17) $$

Some typical solution curves of the logistic equation are shown in Fig. 9.5.

Figure 9.5. Solution curves for $y(t)$ in the scaled form of the logistic equation based on (9.18). We show the predicted behaviour of $y(t)$ as given by Eqn. (9.17) for three different initial conditions, $y_0 = 0.1, 0.25, 0.5$. Note that all solutions approach the value $y = 1$.

9.6.6 What this solution tells us

We have arrived at the function that describes the scaled population as a function of time as predicted by the scaled logistic equation, (9.16). The level of population (in units of the carrying capacity $K$) follows the time-dependent function

$$ y(t) = \frac{y_0}{y_0 + (1 - y_0)e^{-rt}}. \quad (9.18) $$
9.7. Extensions and other population models: the “Law of Mortality”

We can convert this result to an equivalent expression for the unscaled total population $N(t)$ by recalling that $y(t) = N(t)/K$. Substituting this for $y(t)$, and noting that $y_0 = N_0/K$ leads to

$$
N(t) = \frac{N_0}{(N_0 + (K - N_0)e^{-rt})}.
$$

(9.19)

It is left as an exercise for the reader to check this claim.

Now recall that $r > 0$. This means that $e^{-rt}$ is a decreasing function of time. Therefore, (9.18) implies that, after a long time, the term $e^{-rt}$ in the denominator will be negligibly small, and so

$$
y(t) \to \frac{y_0}{y_0} = 1,
$$

so that $y$ will approach the value 1. This means that

$$
\frac{N}{K} \to 1 \quad \text{or simply} \quad N(t) \to K.
$$

The population will thus settle into a constant level, i.e., a steady state, at which no further change will occur.

As an aside, we observe that this too, could have been predicted directly from the differential equation. By setting $dy/dt = 0$, we find that

$$
0 = ry(1 - y),
$$

which suggests that $y = 1$ is a steady state. (This is also true for the less interesting case of no population, i.e. $y = 0$ is also a steady state.) Similarly, this could have been found by setting the derivative to zero in Eqn. (9.15), the original, unscaled logistic differential equation. Doing so leads to

$$
\frac{dN}{dt} = 0 \Rightarrow rN \left(\frac{K - N}{K}\right) = 0.
$$

If $r > 0$, the only values of $N$ satisfying this steady state equation are $N = 0$ or $N = K$. This implies that either $N = 0$ or $N = K$ are steady states. The former is not too interesting. It states the obvious fact that if there is no population, then there can be no population growth. The latter reflects that $N = K$, the carrying capacity, is the population size that will be sustained by the environment.

In summary, we have shown that the behaviour of the logistic equation for population growth is more realistic than the simpler exponential growth we studied earlier. We saw in Figure 9.5, that a small population will grow, but only up to some constant level (the carrying capacity). Integration, and in particular the use of partial fractions allowed us to make a full prediction of the behaviour of the population level as a function of time, given by Eqn. (9.19).

9.7 Extensions and other population models: the “Law of Mortality”

There are many variants of the logistic model that are used to investigate the growth or mortality of a population. Here we extend tools to another example, the gradual decline of
a group of individuals born at the same time. Such a group is called a “cohort”.\textsuperscript{55} In 1825, Gompertz suggested that the rate of mortality, \( m \) would depend on the age of the individuals. Because we consider a group of people who were born at the same time, we can trade “age” for “time”. Essentially, Gompertz assumed that mortality is not constant: it is low at first, and increase as individuals age. Gompertz argued that mortality increases exponentially. This turns out to be equivalent to the assumption that the logarithm of mortality increases linearly with time.\textsuperscript{56} It is easy to see that these two statements are equivalent: Suppose we assume that for some constants \( A > 0, \mu > 0, \)

\[
\ln(m(t)) = A + \mu t. \tag{9.20}
\]

Then Eqn. (9.20) means that

\[
\begin{align*}
\log \text{mortality} \\
\ln(m) \\
A \\
\text{slope } \mu \\
age, t
\end{align*}
\]

Figure 9.6. In the Gompertz Law of Mortality, it is assumed that the log of mortality increases linearly with time, as depicted by Eqn. 9.20 and by the solid curve in this diagram. Here the slope of \( \ln(m) \) versus time (or age) is \( \mu \). For real populations, the mortality looks more like the dashed curve.

\[
m(t) = e^{A+\mu t} = e^A e^{\mu t}
\]

Since \( A \) is constant, so is \( e^A \). For simplicity we define Let us define \( m_0 = e^A \). (\( m_0 = m(0) \) is the so-called “birth mortality” i.e. value of \( m \) at age 0.) Thus, the time-dependent mortality is

\[
m = m(t) = m_0 e^{\mu t}. \tag{9.21}
\]

9.7.1 Aging and Survival curves for a cohort:

We now study a population model having Gompertz mortality, together with the following additional assumptions.

\textsuperscript{55} This section was formulated with help from Lu Fan

\textsuperscript{56} In actual fact, this is likely true for some range of ages. Infant mortality is generally higher than mortality for young children, whereas mortality levels off or even decreases slightly for those oldest old who have survived past the average lifespan.
1. All individuals are assumed to be identical.

2. There is “natural” mortality, but no other type of removal. This means we ignore the mortality caused by epidemics, by violence and by wars.

3. We consider a single cohort, and assume that no new individuals are introduced (e.g. by immigration).

We will now study the size of a “cohort”, i.e. a group of people who were born in the same year. We will denote by $N(t)$ the number of people in this group who are alive at time $t$, where $t$ is time since birth, i.e. age. Let $N(0) = N_0$ be the initial number of individuals in the cohort.

### 9.7.2 Gompertz Model

All the people in the cohort were born at time (age) $t = 0$, and there were $N_0$ of them at that time. That number changes with time due to mortality. Indeed,

$$\text{The rate of change of cohort size} = -\text{[number of deaths per unit time]}$$

$$= -\text{[mortality rate]} \cdot \text{[cohort size]}$$

Translating to mathematical notation, we arrive at the differential equation

$$\frac{dN(t)}{dt} = -m(t)N(t),$$

and using information about the size of the cohort at birth leads to the initial condition, $N(0) = N_0$. Together, this leads to the initial value problem

$$\frac{dN(t)}{dt} = -m(t)N(t), \quad N(0) = N_0.$$ 

Note similarity to Eqn. (9.1), but now mortality is time-dependent.

In the Problem set, we apply separation of variables and integrate over the time interval $[0, T]$ to show that the remaining population at age $t$ is

$$N(t) = N_0 e^{-\frac{m_0}{\mu} (e^{\mu t} - 1)}.$$ 

### 9.8 Summary

In this chapter, we used integration methods to find the analytical solutions to a variety of differential equations where initial values were prescribed.

We investigated a number of population growth models:

1. Exponential growth, given by $\frac{dy}{dt} = ky$, with initial population level $y(0) = y_0$ was investigated (Eqn. (9.1)). This model had an unrealistic feature that growth is unlimited.

---

57Note that new births would contribute to other cohorts.
2. The Logistic equation \( \frac{dN}{dt} = rN \left( \frac{K-N}{K} \right) \) was analyzed (Eqn. (9.15)), showing that density-dependent growth can correct for the above unrealistic feature.

3. The Gompertz equation, \( \frac{dN(t)}{dt} = -m(t)N(t) \), was solved to understand how age-dependent mortality affects a cohort of individuals.

In each of these cases, we used separation of variables to “integrate” the differential equation, and predict the population as a function of time.

We also investigated several other physical models in this chapter, including the velocity of a falling object subject to drag force. This led us to study a differential equation of the form \( \frac{du}{dt} = a - by \). By slight reinterpretation of terms in this equation, we can use results to understand chemical kinetics and blood alcohol levels, as well as a host of other scientific applications.

Section 9.5, the “centerpiece” of this chapter, illustrated the detailed steps that go into the formulation of a differential equation model for flow of liquid out of a container. Here we saw how conservation statements and simplifying assumptions are interpreted together, to arrive at a differential equation model. Such ideas occur in many scientific problems, in chemistry, physics, and biology.