Chapter 4

Applications of the definite integral to rates, velocities and densities

4.1

Two cars, labeled 1 and 2 start side by side and accelerate from rest. Figure 4.1 shows a graph of their velocity functions, with $t$ measured in minutes.

(a) At what time(s) do the cars have equal velocities?

(b) At what time(s) do the cars have equal accelerations?

(c) Which car is ahead after one minute?

(d) Which car is ahead after 3 minutes?

(e) When does one car overtake the other? (Give an approximate answer)
4.2

The speed of a car (km/h) is given by the expression

\[ v(t) = 2t^2 + 5t, \quad 0 < t < 1 \]

where \( t \) is time in hours. Use this expression to find

(a) The acceleration over this time period.

(b) The total displacement over the same time period.

4.3

The velocity of a boat moving through water is found to be \( v(t) = 10(1 - e^{-t}) \).

(a) Find the acceleration of the boat and show that it satisfies a differential equation, i.e an equation of the form \( \frac{d[a(t)]}{dt} = -k \cdot a(t) \) for some value of the constant \( k \) (i.e. find that value of \( k \)).

(b) Find the displacement of the boat at time \( t \).

4.4

A particle in a force field accelerates so that its acceleration is described by the function

\[ a(t) = t - t^2, \quad 0 < t < 1 \]

Find the velocity of the particle \( v(t) \) for \( 0 < t < 1 \) assuming that it starts at rest. Then find the total displacement of the particle over the time interval \( 0 < t < 1 \).

4.5

An express mail truck delivers mail to various companies situated along a central avenue and often goes back and forth as new mail arrives. Over some period of time, \( 0 < t < 10 \), its velocity (in km per hour) can be described by the function: \( v(t) = t^2 - 9t + 14 \) Find:

(a) The displacement over this period of time. [Hint: recall that if you leave home in the morning, travel to work, and then go back home, then your net distance traveled, or total displacement, over this full period of time is zero.]

(b) How much gasoline was consumed during this period of time if the vehicle uses 5 liters per km. [Hint: To answer (b), you will need to find the total distance that the vehicle actually covered during its trip.]
4.6

The reaction time of a driver (time it takes to notice and react to danger in the road ahead) is about 0.5 seconds. When the brakes are applied, it then takes the car some time to decelerate and come to a full stop. If the deceleration rate is \( a = -8 \text{ m/sec}^2 \), how long would it take the driver to stop from an initial speed of 100 km per hour? (Include both reaction time and deceleration time, and use the Fundamental Theorem of Calculus to arrive at your answer.)

4.7

You are driving your car quickly (at speed 100 km/hr) to catch your flight to Hawaii for mid-term break. A pedestrian runs across the road, forcing you to brake hard. Suppose it takes you 1 sec to react to the danger, and that when you apply your brakes, you slow down at the rate \( a = -10 \text{ m/sec}^2 \). How long will it take you to stop? How far will your car move from the instant that the danger is sighted until coming to a complete stop? Use the Fundamental Theorem of Calculus to arrive at your answer.

4.8

The density of a band of protein along a one-dimensional strip of gel in an electrophoresis experiment is given by \( p(x) = 2(x - 1)(2 - x) \) for \( 1 \leq x \leq 2 \), where \( x \) is the distance along the strip in cm and \( p(x) \) is the protein density (i.e. protein mass per cm) at distance \( x \). Graph the density \( p \) as a function of \( x \). Find the total mass of the protein in the band for \( 1 \leq x \leq 2 \). [Hint: simplify the function first.]

4.9

In ancient Egypt, most of the population was confined an area within 0.5km of the river Nile. The density of the population decreased with increasing distance from the sea. Suppose that the population density at distance \( x \) from the mouth of the river was \( p(x) = 200e^{-0.001x} \) people per km, where \( x \) is distance in km. (Note: the width of the populated region is \( w = 1 \text{km} = \text{constant} \), so you need only integrate in the \( x \) direction.)

(a) What was the total population size along the first 100 km of the river?

(b) What was the total population size along the whole length of the river (assume the Nile is infinitely long)?
4.10

The air density $h$ meters above the earth’s surface is

$$p(h) = Ae^{-ah}(kg/m^3)$$

Find the mass of a cylindrical column of air of radius $r = 2$ meters and height $H = 25000$ meters. Let $A = 1.28$ (kg/m$^3$), $a = 0.000124$ per meter. In order to set up the integral, we consider how to approach this as a Riemann Sum.

4.11

A test-tube contains a solution of glucose which has been prepared so that the concentration of glucose is greatest at the bottom and decreases gradually towards the top of the fluid. (This is called a density gradient). Suppose that the concentration $c$ as a function of the depth $x$ is $c(x) = x/10$ (in units of gm/cm$^3$). The radius of the tube is 2 cm and the height of the glucose-containing solution is 10 cm. Determine the total amount of glucose in the tube (in gm).

4.12

The growth rate of a pair of twins (in gm/day) is shown in Figure 4.2. Suppose both children have the same weight at time $t = 0$.

(a) Which one is larger at age 0.5 years?

(b) Which one is larger at age 1 year?
4.13

The flow rate of blood through the heart can be described approximately as a periodic function of the form \( F(t) = A(1 + \sin(0.15t)) \), where \( t \) is time in seconds and \( A \) is a constant in units of cubic cm per second. (Thus, at time \( t \), \( F(t) \) cubic cm of blood flow through the heart per second.) Find the total volume of blood that flows through the heart between \( t = 0 \) and \( t = 1 \). (Express your answer in terms of \( A \).)

4.14

During a gambling session lasting 6 hours, the rate of winnings at a casino in dollars per hour are seen to follow the formula \( w(t) = 2000t(1 - (t/6)) \). Find the total winnings during that whole session.

4.15

At time \( t \), an intravenous infusion delivers a flow rate of \( y = 100(1 - t^3) \text{ cm}^3/\text{hr} \), where \( t \) is time in hours. The infusion contains a drug at concentration 0.1 \( \text{mg/cm}^3 \). Find the total volume of fluid and the total amount of drug delivered to the patient over a 1 hour period from \( t = 0 \) to \( t = 1 \).

4.16

Oil leaks out of an oil tanker at the rate \( f(t) = 10 - 0.2t^2 \) (where \( f \) is in 10 thousand barrels per hour and \( t \) is in hours). (Note: This function only makes sense as long as \( f(t) \geq 0 \) since a negative flow of oil is meaningless in this case.)

(a) At what time will the flow be zero?
(b) What is the total amount that has leaked out between \( t = 0 \) and the time you found in (a)?

4.17

After a heavy rainfall, the rate of flow of water into a lake is found to satisfy the relationship \( F(t) = 4 - \left( \frac{t}{10} - 1 \right)^2 \) where \( t \) is time in hours, \( 0 \leq t \leq 30 \) and \( F \) is in units of 100,000 gallons/hr.

(a) Find the time, \( t_1 \) at which the rate of flow is greatest.
(b) Find the time, \( t_2 \) at which the flow is zero.
(c) Find how much water in total has flowed into the lake between these two times.
4.18

The growth rate of a crop is known to depend on temperature during the growing season. Suppose the growth rate of the crop in tons per day is given by 

\[ g(t) = 0.1(T(t) - 18) \]

where \( T(t) \) is temperature in degrees Celsius. Suppose the temperature record during the 90 days of the season was \( T(t) = 22 + 0.1t + 4\sin(2\pi t/60) \) where \( t \) is time in days. Find the total growth (in tons) that would have occurred over the whole season.

4.19

This question concerns cumulative exposure to radiation experienced by people living near nuclear waste disposal sites.

(a) Recall from last term that radioactive material decays according to a negative exponential:

\[ m(t) = m_0 \cdot e^{-rt} \]

where \( m(t) \) is the mass of the radioactive material at time \( t \), \( m_0 = m(0) \) is the initial mass at time \( t = 0 \), and \( r \) is the rate of decay. The half-life is the time it takes for the material to decay to one half its initial mass. Suppose that \( t \) is measured in months. Determine the rate of decay \( r \) if the half-life is 1 month.

(b) Assume that at any time, the amount of radiation is proportional to the mass of the radioactive material. If initially the radiation level is 0.5 rads per month, how could we describe the radiation level as a function of time?

(c) Assume now that there is radioactive material in your backyard of the type considered in (a) above, i.e. that it has a half-life of one month. Calculate the cumulative exposure in rads that would occur if you lived in your house for 10 years.

4.20

The level of glucose in the blood depends on the rate of intake from ingestion of food and on the rate of clearance due to glucose metabolism. Shown in Figure 4.3 are two functions, \( I(t) \) and \( C(t) \) for the intake and clearance rates over a period of time after fasting. Both rates are functions of time \( t \). Suppose that at time 0 there is no glucose in the blood.

(a) Express the level of blood glucose as a definite integral.

(b) At what approximate time was the intake rate maximal?

(c) At what approximate time was the clearance rate maximal?

(d) When was the blood level of glucose maximal?
4.21

The rate at which water flows in and out of Capilano Reservoir is described by two functions. \( I(t) \) is the rate at which water flows into the reservoir (in gallons per day) and \( O(t) \) is the rate at which water flows out (in gallons per day). See sketch below in Figure 4.4. Assume that there is water in the reservoir at time \( t = 0 \).

(a) Express the quantity of water \( Q(t) \) in the reservoir as a definite integral. (i.e. \( Q(0) > 0 \)).

(b) When is the quantity of water in the reservoir greatest and when is it smallest?

4.22

The rate at which animals migrate into and out of a wildlife reserve is described by two functions shown in Figure 4.5. \( I(t) \) is the rate at which animals enter the reserve and \( O(t) \) is the rate at which they leave (both in number per day).
Figure 4.5: For problem 4.22

(a) Express the number of animals in the reserve as a definite integral.

(b) When is the number of animals in the reserve greatest and when is it smallest?

4.23

During a particularly soggy week in Vancouver, rainfall reached epic proportions. The rainfall pattern was as follows: A constant 20mm over the first day, a steady increase from 20 up to 50 mm over the next day (assume linear increase), 50mm rain over the next day, a steady drop from 50 down to 40 mm over the next day, and a flat 30 mm over the next day. Determine the total amount of rain during this period and the average daily rainfall for the same period.

4.24  [98 Final]

Find the average value of the function $f(x) = \sin(\frac{\pi x}{2})$ over the interval [0,2].

4.25

(a) Find the average value of $x^n$ over the interval [0,1].

(b) What happens as $n$ becomes arbitrarily large (that is, $n \to \infty$)?

(c) Explain your answer to part (b) by considering the graphs of these functions.

(d) Repeat parts (a) - (c) using the functions $x^{1/n}$.
4.26

An object starts from rest at \( t = 0 \) and accelerates so that \( a = \frac{dv}{dt} = ce^{-t} \) where \( c \) is a positive constant.

(a) Find the average velocity over the first \( t \) seconds.

(b) What happens to this average velocity as \( t \) becomes very large?

4.27 Symmetry

(a) Find the average value of the function \( \sin x \) over the interval \([-\pi, \pi]\).

(b) Find the average value of the function \( x^3 \) over the interval \([-1, 1]\).

(c) Find the average value of the function \( x^3 - x \) over the interval \([-1, 1]\).

(d) Explain these results graphically.

(e) Find the average value of an odd function \( f(x) \) over the interval \([-a, a]\). (Remember that \( f(x) \) is odd if \( f(-x) = -f(x) \).)

(f) Suppose now that \( f(x) \) is an even function (that is, \( f(-x) = f(x) \)) and its average value over the interval \([0, 1]\) is 2. Find its average value over the interval \([-1, 1]\).

4.28

The intensity of light cast by a street lamp at a distance \( x \) (in meters) along the street from the base of the lamp is found to be approximately \( I(x) = 400 - x^2 \) in arbitrary units for \(-20 < x < 20\).

(a) Find the average intensity of the light over the interval \(-5 < x < 5\).

(b) Find the average intensity over \(-7 < x < 7\).

(c) Find the value of \( b \) such that the average intensity over \([-b, b]\) is \( I_{av} = 10\).

4.29 Rates of hormone production and removal

Consider the rate of hormone production \( p(t) \) and the rate of removal \( r(t) \) given by

\[ p(t) = A(1 + \sin(\omega t)), \quad r(t) = A(1 + \cos(\omega t)). \]

for \( \omega = \pi/12 \). Calculate the net increase in hormone over the time period \( 3 \leq t \leq 15 \) (in hours).
4.30

The length of time from sunrise to sunset (in hours) $t$ days after the Spring Equinox is given by

$$l(t) = 12 + 4 \sin \left( \frac{\pi t}{182} \right)$$

(a) Explain the meaning of the word “Equinox” and describe what happens on that day according to the above formula.

(b) What is the length of the shortest and the longest day and when do these occur according to this formula?

(c) How long is one complete cycle in this expression?

(d) Sketch $l(t)$ as a function of $t$.

(e) Find the average day-length over the month immediately following the Equinox.

(f) Find the average day length over the whole year. Explain your result with a simple geometric or intuitive argument.

4.31

Consider the periodic function,

$$f(t) = \sin(2t) + \cos(2t)$$

(a) What is the frequency, the amplitude, and the length of one cycle in this function? (You are asked to express $f(t)$ in the form $A \sin(\omega t + \varphi)$. Then we use the terminology $A =$ amplitude, $\varphi =$ phase shift, $\omega =$ frequency.

(b) How would you define the average value of this function over one cycle?

(c) Compute this average value and show that it is zero. Now explain why this is true using a geometric argument.

4.32

The current in an AC electric circuit is given by

$$I(t) = A \cos(\omega t)$$

The power in the circuit is defined as $P(t) = I^2(t)$.

(a) What is meant by one cycle in this situation?
(b) Sketch graphs of $I(t)$ and $P(t)$. Explain why $P(t)$ is always positive, and indicate how its zeros are related to zeros of $I(t)$. What are the maximal and minimal values of each of these functions?

(c) Find the average power and the average current over half a cycle. (Note: in computing the average power, you will need to use the trick $\cos^2(\omega t) = \frac{1}{2}(1 + \cos(2\omega t))$.)