Chapter 5

Applications of the definite integral to calculating volume, mass, center of mass, and length

5.1

Five beads are distributed along a thin 1 dimensional wire. Their masses and positions are: \( m_1 = 5, x_1 = 0; m_2 = 2, x_2 = 3; m_3 = 1, x_3 = 4; m_4 = 2, x_4 = 5; m_5 = 10, x_5 = 6 \). Find the total mass and the center of mass of this discrete mass distribution.

5.2

Suppose that the function \( \rho(x) \) represents the density of a bar for \( a \leq x \leq b \). Explain the distinction between:

(a) the average density of the bar, \( \bar{\rho} \)

(b) the mass of the bar, \( M \), and

(c) the center of mass (or centroid) of the bar, \( \bar{x} \).

5.3

The density of a bar is given by

\[ \rho(x) = 1 - x \quad 0 < x < 1 \]

(a) Sketch the density of the bar and the function

\[ M(x) = \int_0^x p(s) \, ds. \]
Find the total mass of the bar.

(b) Find the average mass density along the bar.

(c) Find the center of mass of the bar.

(d) Where along the length of the bar should you cut to get two pieces of equal mass?

(e) What fraction of the mass of the bar is found between \( x = 0 \) and \( x = 0.5 \)?

5.4

The density of a bar is given by
\[
\rho(x) = ax^2 \quad 0 < x < L
\]

(a) Find the total mass of the bar.

(b) Find the average mass density along the bar.

(c) Find the center of mass of the bar.

(d) Where along the length of the bar should you cut to get two pieces of equal mass?

5.5

The density of a beam is given by the function \( \rho(x) = x^{m/n} \) where \( 0 \leq x \leq 1 \).

(a) Find the center of mass \( \bar{x} \).

(b) Explain what happens to the center of mass if \( m \) is very large (for \( n = 1 \)).

(c) What happens to the center of mass if \( n \) is very large, (for \( m = 1 \))?  

5.6 Density of cars along a highway

At rush hour, the density of cars along a highway is given by
\[
C(x) = 100x \left(1 - \frac{x}{10}\right) \quad 0 \leq x \leq L,
\]

where \( x \) is distance in kilometers.

(a) What is the largest value of \( L \) for which this density makes sense?

(b) Where along the highway is the congestion greatest? What is the car density at that location?

(c) What is the total number of cars along the road?
5.7

To investigate changes in the Earth’s weather, scientists examine the distribution of pollen grains in a 1 dimensional drilled “core sample”, i.e. a sample of the Earth’s crust that contains archaeological deposits of soil from many thousands of years. Assume that the core sample is a cylinder of unit cross-sectional area and length \( L \). Suppose that pollen grain density \( p(x) \) at a point \( x \) in this sample is in a core sample of length \( L \) is given by

\[
p(x) = A \sin(ax), \quad 0 < x < L = \frac{\pi}{a}.
\]

where \( p(x) \) are the number of particles per unit volume at a distance \( x \) from one end of the sample.

(a) Where is the pollen grain most concentrated along this one dimensional sample?

(b) Find the average density of pollen grains along the length of the sample.

(c) Find the center of mass of the pollen grain distribution. (Note: you can use the fact that the density is distributed symmetrically to avoid having to integrate.)

5.8

Find the volume of a cone whose height \( h \) is equal to its base radius \( r \), by using the disc method. We will place the cone on its side, as shown in the Figure 5.1, and let \( x \) represent position along its axis.

(a) Using the diagram shown below (Figure 5.1), explain what kind of a curve in the xy plane we would use to generate the surface of the cone as a surface of revolution.

![Figure 5.1: For problem 5.8](image)

(b) Using the proportions given in the problem, specify the exact function \( y = f(x) \) that we need to describe this “curve”.

(c) Now find the volume enclosed by this surface of revolution for \( 0 \leq x \leq 1 \).

(d) Show that, in this particular case, we would have gotten the same geometric object, and also the same enclosed volume, if we had rotated the “curve” about the y axis.
5.9

Find the volume of the cone generated by revolving the curve \( y = f(x) = 1 - x \) (for \( 0 < x < 1 \)) about the \( y \) axis. Use the disk method, with disks stacked up along the \( y \) axis.

5.10

Find the volume of the “bowl” obtained by rotating the curve \( y = 4x^2 \) about the \( y \) axis for \( 0 \leq x \leq 1 \).

5.11

On his wedding day, Kepler wanted to calculate the amount of wine contained inside a wine barrel whose shape is shown below in Figure 5.2. Use the disk method to compute this volume. You may assume that the function that generates the shape of the barrel (as a surface of revolution) is \( y = f(x) = R - px^2 \), for \(-1 < x < 1\) where \( R \) is the radius of the widest part of the barrel. (\( R \) and \( p \) are both positive constants.)

![Figure 5.2: For problem 5.11](image)

5.12

Consider the curve

\[
y = f(x) = 1 - x^2 \quad 0 < x < 1
\]

rotated about the \( y \) axis. Recall that this will form a shape called a paraboloid. Use the cylindrical shell method to calculate the volume of this shape. Note: this technique is optional. See the Appendix of course notes for an example of the shell method.
5.13

Find the volume of the solid obtained by rotating the region bounded by the given curves $f(x)$ and $g(x)$ about the specified line.

(a) $f(x) = \sqrt{x} - 1$, $g(x) = 0$, from $x = 2$ to $x = 5$, about the $x$-axis.
(b) $f(x) = \sqrt{x}$, $g(x) = x/2$, about the $y$-axis.
(c) $f(x) = 1/x$, $g(x) = x^3$, from $x = 1/10$ to $x = 1$, about the $x$-axis.

5.14

Let $R$ be the region contained between $y = \sin(x)$ and $y = \cos(x)$ for $0 \leq x \leq \pi/2$. Write down the expression for the volume obtained by rotating $R$ about

(a) $x$-axis; and
(b) the line $y = -1$.

Do not integrate.

5.15

Suppose a lake has a depth of 40 meters at its deepest point and is bowl-shaped, with the surface of the bowl generated by rotating the curve $z = x^2/10$ around the $z$-axis. Here $z$ is the height in meters above the lowest point of the bowl. The distribution of sediment in the lake is stratified by height along the water column. In other words, the density of sediment (in mass per unit volume) is a function of the form $s(z) = C(40 - z)$, where $z$ is again vertical height in meters from the point at the bottom of the lake. Find the total mass of sediment in the lake (Your answer will have the constant $C$ in it.). The volume of the lake is the volume above the curve $z = x^2/10$ and below $z = 40$.

5.16

In this problem you are asked to find the volume of a height $h$ pyramid with a square base of width $w$. (This is related to the Cheops pyramid problem, but we will use calculus.) Let the variable $z$ stand for distance down the axis of the pyramid with $z = 0$ at the top, and consider “slicing” the pyramid along this axis (into horizontal slices). This will produce square “slices” (having area $A(z)$ and some width $\Delta z$). Calculate the volume of the pyramid as an integral by figuring out how $A(z)$ depends on $z$ and integrating this function.
5.17

Set up the integral that represents the length of the following curves: Do not attempt to calculate the integral in any of these cases

(a)  
\[ y = f(x) = \sin(x) \quad 0 < x < 2\pi. \]

(b)  
\[ y = f(x) = \sqrt{x} \quad 0 < x < 1. \]

(c)  
\[ y = f(x) = x^n \quad -1 < x < 1. \]

5.18

Compute the length of the line \( y = 2x + 1 \) for \(-1 < x < 1\) using the arc-length formula. Check your work by using the simple distance formula (or Pythagorean theorem).

5.19

Second Spreadsheet assignment: You are told that the derivative of a certain function is

\[ f'(x) = 1 - 2 \sin^2\left(\frac{x}{3}\right) \]

and that \( f(0) = 0 \). Use the spreadsheet to create one plot that contains all of the following graphs:

(1) The graph of the function \( y = f(x) \) (whose derivative is given to you). This should be plotted over the interval from 0 to 9.

(2) The graph of the “element of arclength” \( dl = \sqrt{1 + (f'(x))^2} \, dx \) showing how this varies across the same interval.

(3) The graph of the cumulative length of the curve \( L(x) \). Briefly indicate what you did to find the function in part (1). (You might consider how the spreadsheet would help you calculate the values of the desired function, \( y = f(x) \), rather than trying to find an expression for it.)
5.20  work 1

A spring has a natural length of 16 cm. When it is stretched \( x \) cm beyond that, Hooke’s Law states that the spring pulls back with a restoring force \( F = kx \) dyne, where the constant \( k \) is called the spring constant, and represents the stiffness of the spring. For the given spring, 8 dyne of force are required to hold it stretched by 2 cm. How much work (dyne-cm) is done in stretching this spring from its natural length to a length 24 cm? (Note: use integration to set up this problem.)

5.21  work 2

Calculate the work done in pumping water out of a parabolic container up to the height \( h = 10 \) units. Assume that the container is a surface of revolution generated by rotating the curve \( y = x^2 \) about the \( y \) axis, that the height of the water in the container is 10 units, that the density of water is 1gm/cm\(^3\) and that the force due to gravity is \( F = mg \) where \( m \) is mass and \( g = 9.8 m/s^2 \).