Chapter 6

Techniques of Integration

6.1 Differential Notation 1

Calculate the differential of the following functions by using the definition

\[ dy = y'(x)dx. \]

Express the result in terms of the product between \( y'(x) \) and the differential of \( x, \, dx \).

For example, given

\[ y(x) = 3x + \sin(2x), \]

its differential is

\[ dy = y'(x)dx = (3 + 2\cos(2x))dx. \]

- (a) \( f(x) = e^{x^2} \)
- (b) \( f(x) = (x + 1)^2 \)
- (c) \( f(x) = \sqrt{x} \)
- (d) \( f(x) = \arcsin(x) \)
- (e) \( f(x) = x^2 + 3x + 1 \)
- (f) \( f(x) = \cos(2x) \)
- (g) \( y = x^6 + 2x^4 - 2x \)
- (h) \( y = (x - 2)^2(x + 1)^5 \)
- (i) \( y = x/(x + 3) \)

6.2 Differential Notation 2

Given the differential of a function \( y \) in terms of the product between its derivative \( y'(x) \) and the differential of \( x, \, dx \), express the same differential in terms of the the differential of \( y \) itself, i.e., \( dy \). Then, use the result combined with the fundamental theorem of calculus to calculate the corresponding indefinite integrals.

For example, if we know that the differential of a function \( y(x) \) is given by

\[ (3 + 2\sin(2x))dx = ?, \quad (\text{Or equivalently, } \int (3 + 2\sin(2x))dx = ?) \]

we now need to figure out that \( y(x) = 3x - \cos(2x) \) and express the differential given above in terms of the differential of \( y \) itself

\[ (3 + 2\sin(2x))dx = d(3x - \cos(2x)). \]
Using this result and the fundamental theorem of calculus, we can solve the following

\[ \int (3 + 2\sin(2x))\,dx = \int d(3x - \cos(2x)) = 3x - \cos(2x) + C. \]

(Those marked by a star are a little bit more difficult since they involve composite functions.)

(a) \(x^3\,dx = ?\), \(\int x^3\,dx = ?\)  
(b) \(x^{3/2}\,dx = ?\), \(\int x^{3/2}\,dx = ?\)

c) \(\frac{1}{x^2}\,dx = ?\), \(\int \frac{1}{x^2}\,dx = ?\)  
(d) \(\sqrt{x}\,dx = ?\), \(\int \sqrt{x}\,dx = ?\)

e) \(\frac{1}{\sqrt{x}}\,dx = ?\), \(\int \frac{1}{\sqrt{x}}\,dx = ?\)  
(f) \(e^{-2x}\,dx = ?\), \(\int e^{-2x}\,dx = ?\)

(g) \((x + 1)^2\,dx = ?\), \(\int (x + 1)^2\,dx = ?\)  
(h) \(\frac{1}{(x+3)^2}\,dx = ?\), \(\int \frac{1}{(x+3)^2}\,dx = ?\)

(i) \(\cos(2x)\,dx = ?\), \(\int \cos(2x)\,dx = ?\)  
(j) \(\sec^2(x)\,dx = ?\), \(\int \sec^2(x)\,dx = ?\)

(k) \(\frac{2}{1+x^2}\,dx = ?\), \(\int \frac{2}{1+x^2}\,dx = ?\)  
(l) \(\frac{3}{1+e^x}\,dx = ?\), \(\int \frac{3}{1+e^x}\,dx = ?\)

6.3 Substitution 1

(a) Evaluate the following indefinite integrals using the substitution method:

\[ \int \sin(3x)\,dx, \quad \int \cos\left(\frac{\sqrt{x}}{x}\right)\,dx, \quad \int x^3\sqrt{x^4 + 1}\,dx, \quad \int \frac{4}{1 + 2x}\,dx. \]

(b) Calculate the following definite integrals using the substitution rule:

\[ \int_0^5 (\sqrt{3 + 2x})\,dx, \quad \int_0^{\pi/4} \sin(4t)\,dt, \quad \int_0^{\pi} x\cos(x^2)\,dx. \]

6.4 Substitution 2

Compute the following integrals using substitution.

(a) \(\int_1^p \frac{1}{1+y^2}\,dy\)  
(b) \(\int x^2(x^3 + 1)^6\,dx\)  
(c) \(\int e^x\sqrt{1+2e^x}\,dx\)

d) \(\int \frac{1}{x \ln(x)}\,dx\)  
(e) \(\int \frac{1}{1-y}\,dy\)  
(f) \(\int_1^k \frac{k_1}{k_1 - n}\,dn, \quad k_1, k_2 > 0\)

g) \(\int \frac{x^2}{1-x^3}\,dx\)  
(h) \(\int \sqrt{3x+1}\,dx\)  
(i) \(\int \frac{x}{\sqrt{4+x^2}}\,dx\)

(j) \(\int \cos(x)\sin^5(x)\,dx\)  
(k) \(\int \frac{3}{4+5x}\,dx\)  
(l) \(\int \cot(\theta)\,d\theta\)

(m) \(\int \frac{\sec^2(x)}{\sqrt{2+\tan(x)}}\,dx\)  
(n) \(\int \frac{2}{4+x^2}\,dx\)
6.5 Trigonometric Substitution 1

The integral \( \int \sin(x) \cos(x) \, dx \) can be done in several ways:

(a) using the substitution \( u = \sin(x) \)

(b) by first using the trigonometric identity \( \sin(2x) = 2 \sin(x) \cos(x) \) and then integrating. Show that the two answers are equivalent. (Hint: you will find that this is a good opportunity to review trigonometric identities.)

Note: Part (a) could also be done by the substitution \( u = \cos(x) \).

6.6 Trigonometric substitution 2

The integral \( \int_0^1 \sqrt{1 - x^2} \, dx \) can be done by making the substitution \( x = \sin(u) \) and \( dx = \cos(u) \, du \). This is called a trigonometric substitution.

(a) Show that this reduces the integral to \( \int \cos^2(u) \, du \).

(b) Now use the identity \( \cos^2(u) = \frac{1 + \cos(2u)}{2} \) to re-express the integral in a simpler form. Then integrate.

(c) Explain why the answer is the same as the area of 1/4 of a circle of radius 1.

6.7 Partial Fractions 1

Practice with Integration: Compute the following integrals. Use factoring and/or completing the square and partial fractions, or some other technique if necessary.

(a) \( \int \frac{1}{x^2 - x - 20} \, dx \)

(b) \( \int \frac{3}{x^2 + 6x + 9} \, dx \)

(c) \( \int \frac{-1}{x^2 + 4x + 14} \, dx \)

(d) \( \int \frac{2}{x^2 - 6x + 8} \, dx \)
6.8 Partial Fraction 2

The integrals shown below may look very similar, but in fact they lead to quite different results:

(a) \[ \int \frac{dx}{a^2 + x^2} \]

(b) \[ \int \frac{dx}{a^2 - x^2} \]

Show that (a) can be reduced to an inverse tangent type integral by a bit of algebraic rearrangement and a substitution of the form \( u = x/a \). Show that (b) can be integrated by factoring the expression \( a^2 - x^2 \) and using the method of partial fractions.

6.9 Integration by Parts 1

Use integration by parts to show that \( \int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx \)

6.10 Integration by Parts 2

First make a substitution and then use integration by parts to evaluate the following integrals:

(a) \[ \int \sin(\sqrt{x}) \, dx \]

(b) \[ \int_{1}^{4} e^{\sqrt{x}} \, dx \]

(c) \[ \int_{\sqrt{\pi/2}}^{\sqrt{\pi}} x^3 \cos(x^2) \, dx \]

(d) \[ \int x^5 e^{x^2} \, dx \]
6.11

Compute the following integrals.

(a) \[\int \frac{3}{x^2 + 4x + 6} \, dx\]

(b) \[\int \frac{2}{(x+1)(x+2)} \, dx\]

(c) \[\int \frac{1}{x^2 + 6x + 8} \, dx\]

(d) \[\int_1^p \frac{1}{1 - y^2} \, dy\]

(e) \[\int_0^T te^{-2t} \, dt\]

(f) \[\int_0^\pi x \sin\left(\frac{x}{2}\right) \, dx\]

(g) \[\int x^2 \ln(x) \, dx\]

6.12

Integration drills: (Note: some simplifications in a few of these will make your work much easier.)

(1) \[\int \frac{1}{x} \ln(x) \, dx\]

(2) \[\int \frac{2}{2 + 3x} \, dx\]

(3) \[\int x^2 \sin(3x^3 + 1) \, dx\]

(4) \[\int_1^2 2t e^{3t^2} \, dt\]

(5) \[\int_0^1 \frac{x}{1 + x^2} \, dx\]

(6) \[\int \frac{1}{\sqrt{2 + 3x^2}} \, dx\]

(7) \[\int \frac{2}{3 + 4x^2} \, dx\]

(8) \[\int \frac{1}{\sqrt{25 - 4x^2}} \, dx\]

(9) \[\int x \frac{1}{\sqrt{9 + x^2}} \, dx\]

(10) \[\int \cos(3x) (1 - \sin(3x)) \, dx\]

(11) \[\int \cos(2x)(1 - \sin(2x)) \, dx\]

(12) \[\int \sec^2(x) \sqrt{\tan(x) + 1} \, dx\]

(13) \[\int_0^{\pi/2} (1 - \sin^2(t)) \sin(t) \, dt\]

(14) \[\int 2x(1 - \sin(2x)) \, dx\]

(15) \[\int \sec(x) \tan(x) \, dx \text{ (Hint: use your trig relations)}\]

(16) \[\int x \sin(x + 1) \, dx\]

(17) \[\int (x \sin^2(x) + x \cos^2(x))^5 \, dx \text{ (Hint: look carefully!)}\]

(18) \[\int \sec^2(t) \, dt\]

(19) \[\int \frac{1}{2x^2 + 12x + 18} \, dx \text{ (Some algebra, please)}\]

(20) \[\int \frac{1}{(x - 5)(x + 1)} \, dx\]

(21) \[\int \frac{1}{(x + 3)(2x - 1)} \, dx\]

(22) \[\int \frac{1}{(x + 2)(x + 3)} \, dx\]

(23) \[\int \frac{3x^2 + 1}{x^2(x^2 + 1)^2} \, dx\]

(24) \[\int \frac{1}{x^2 - 3x + 2} \, dx \text{ (Hint: try factoring)}\]

(25) \[\int \frac{3}{x^2 + 3x + 15} \, dx\]
(26) \[ \int x \sec^2(x) \, dx \] (27) \[ \int xe^{x^2} \, dx \] (28) \[ \int x^2e^x \, dx \]

(29) \[ \int \tan^{-1}(x) \, dx \] (30) \[ \int x \cos(x) e^x \, dx \]

### 6.13

Consider the curve \( y = f(x) = \sqrt{1 - x^2} \) (which is a part of a circle of radius 1) over the interval \( 0 < x < 1 \). Suppose this curve is rotated about the \( y \) axis to generate the top half of a sphere. Set up an integral which computes the volume of this hemisphere using the shell method, and compute the volume. (You will need to make a substitution to simplify the integral. To do this question, some techniques of anti-differentiation are required.)

### 6.14

Let the density of mass along a bar of length \( L \) be given by

\[ p(x) = e^{-ax} \]

(a) Find the total mass of the bar.

(b) Find the average mass density along the bar.

(c) Find the center of mass of the bar.

### 6.15

Find the center of mass of a distribution

\[ p(x) = \sin(2x) \quad 0 < x < \frac{\pi}{2} \]

### 6.16

Bacteria are grown in a 1 dimensional tube. The mass density of the bacteria per unit distance along the tube is given by the function

\[ b(x) = \beta \sqrt{a^2 - x^2}, \quad 0 < x < a \]

(a) Find the total mass of bacteria in the tube.

(b) Find the center of mass of the bacteria.
6.17

Gel electrophoresis is an experimental technique used in molecular biology to separate proteins (or other molecules) according to their molecular weights and charges. Suppose that in such an experiment, the distribution of protein along a 1 dimensional strip of this gel is found to be \( p(x) = xe^{-x/2} \) where \( 0 \leq x \leq 5 \) is distance in cm from the end of the strip and \( p(x) \), is the density of the protein per unit distance (in arbitrary units).

(a) Sketch a graph of this distribution.

(b) Determine the location \( x = x_c \) where the density of the protein is greatest. (Indicate on the graph and find using calculus.)

(c) Find the total amount of protein in the region \( x_c - 1 \leq x \leq x_c + 1 \).

(d) Find the mean (i.e. center of mass) of the distribution, i.e. the average \( x \) coordinate of the protein distribution.

6.18

Find the length of the curve

\[ y = f(x) = x^{3/2} \quad 0 < x < 1 \]

(To do this question, some techniques of anti-differentiation are required.)

6.19

To do this question, some techniques of anti-differentiation are required.

(a) Use the chain rule to show that \( F'(x) = \sec(x) \) is the derivative of the function

\[ F(x) = \ln(\sec(x) + \tan(x)) \]

So, what is the anti-derivative of \( f(x) = \sec(x) \)?

(b) Use integration by parts to show that

\[ \int \sec^3(x)dx = \frac{1}{2}(\sec(x) \tan(x) + \ln(\sec(x) + \tan(x))) + C. \]

You will want to recall the trigonometric identity

\[ 1 + \tan^2(x) = \sec^2(x). \]

(c) Set up an integral to compute the arclength of the curve \( y = x^2/2 \) for \( 0 < x < 1 \).

(d) Use the substitution \( x = \tan(u) \) to reduce this arclength integral to an integral of the form

\[ \int \sec^3(u) \, du. \]
6.20 Challenge: Volume of a torus

Find the volume of a solid torus (donut shaped region) with radii $r$ and $R$ as shown in Figure 6.1. (Hint: There are several ways to do this. You can consider this as a surface of revolution and slice it up into little disks with holes (“washers”) as shown.)

![Diagram of a torus and washer](image)

Figure 6.1: For problem 6.20

6.21 Surface Area

Calculate the surface area of a cone shaped surface obtained by rotating the curve $y = \sqrt{x}$ on the interval $[0, 2]$ around the $x$-axis. Use the formula for surface area:

$$S = \int_{0}^{2} 2\pi y(x) \sqrt{1 + (y'(x))^2} \, dx.$$  

To do this question, some techniques of anti-differentiation are required.